International Journal of Theoretical Physics, Vol. 36, No. 3, 1997

Projective Fourier Duality and Weyl Quantization

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Received August 1, 1996

The Weyl–Wigner correspondence prescription, which makes great use of Fourier duality, is reexamined from the point of view of Kac algebras, the most general background for noncommutative Fourier analysis allowing for that property. It is shown how the standard Kac structure has to be extended in order to accommodate the physical requirements. Both an Abelian and a symmetric *projective Kac algebra* are shown to provide, in close parallel to the standard case, a new dual framework and a well-defined notion of *projective Fourier duality* for the group of translations on the plane. The Weyl formula arises naturally as an irreducible component of the duality mapping between these projective algebras.

1. INTRODUCTION

In its broadest meaning, the word "quantization" signifies the passage from the classical to the quantum description of a system. Since the most complete classical description is to be found in the Hamiltonian formalism, the natural path to take is quantization on phase space. This is the main appeal of the Weyl–Wigner approach, which realizes the correspondence principle by attributing a quantum operator to each classical dynamical variable via a Fourier transformation of its density. Conversely, it also attributes a c-number function to each operator by another Fourier transformation, this time involving an integration on operator space. All this supposes the possibility of performing two-way Fourier transformations, that is, of doing a transformation *and its inverse*. Two points should be noted: (i) the integration over operator space, as usually presented, is purely formal and should be

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0020-7748/97/0300-0573$12.50/0

better defined; (ii) the two-way Fourier transformations make use of a deep property of harmonic analysis, Fourier duality. This property only holds under very severe conditions. Actually, at least as usually presented, the Weyl–Wigner formalism supposes a very particular kind of that duality, the Pontryagin duality, which should not be expected to be at work when the phase space is not, for each degree of freedom, the plane $\mathbb{R}^2$ which models vector phase spaces. Pontryagin duality is valid only when the group of linear symplectomorphisms (transformations preserving the phase-space symplectic structure) is Abelian. This group is, for $\mathbb{R}^2$, the translation group $\mathbb{R}^2$. To quantize on more general phase spaces, we need to consider the general approach to Fourier transformations, which requires Kac algebras. We have shown elsewhere (Aldrovandi and Saeger, 1996) how this general formalism, in all its complexity, is necessary even for the simplest nontrivial phase space, the half-plane. We intend here to revisit the apparently well-known $\mathbb{R}^2$ case from this point of view. Because of its vector space structure, the plane would seem not to need a more involved treatment. We shall see that this is not so. It actually conceals a great deal of structure under the appearance of simplicity and, due to its nontrivial cohomology, requires an extension of the very concept of Kac algebra. Furthermore, through the pioneering work of Segal (1953) and the subsequent introduction of weights in the 1960s, the general approach provides a precise meaning to the otherwise mysterious integration over operator space.

Kac algebras are the most general structures presently known on which Fourier analysis can be realized in its integrity. Their rather involved axioms are essential to the most demanding of the properties attached to harmonic analysis, precisely the duality mentioned above. As soon as we depart from the case of functions on Abelian groups, for which the Pontryagin group-to-group duality holds in all its simplicity, Fourier transforms and their inverses can be defined only for functions on domain spaces much more sophisticated than groups. Starting from functions on groups which are separable and locally compact, we arrive necessarily at Kac algebras, which are Hopf–von Neumann algebras endowed with Haar weights. This means that they are noncommutative spaces on which we know how to perform (noncommutative) integration. Roughly speaking, whenever we do Fourier analysis, we are supposing the presence (explicit or not) of Kac algebras. We propose here to bring to light the algebras behind the apparently simple case of the plane $\mathbb{R}^2$. An important point is that, as they are known today, Kac algebras are related to linear representations and as such they are not sophisticated enough to cope with the problem. In order to apply to quantum mechanics, the Kac structure may require an extension to projective representations, and this is precisely what happens in the usual Weyl–Wigner formalism. The situation is rather curious. On one hand, so much is "degenerated" in this simplest of all cases (the space dual to R^2 is R^2 itself, which coincides also with the group manifold of linear symplectomorphisms) that we get the impression that the intricacies of the general formalism can be overlooked; on the other hand, because of its nontrivial cohomology, it requires an extension to projective representations, which is not necessary in other, more complicated, situations. For example, no extension is required when the phase space is the half-plane (Aldrovandi and Saeger, 1996). The central extension of R^2 is, roughly speaking, the Heisenberg group H_3. No extension is a particular case: it can be seen as a trivial extension. We can say that extensions, trivial or not, are required in the generic case and their effects on the standard structures have to be studied.

The essential notation is introduced in Sections 2 and 3, which sum up the usual lore on quantization on phase space and the Heisenberg group. Projective representations of a group can be obtained from the linear representations of its extension. We thus arrive at the projective representations of R^2 from the linear representations of H_3. For Kac algebras, a parallel procedure will be used: we start from the well-established Kac algebra duality for H_3 and then proceed to find the projective Kac algebras of R^2. Actually, a pair of Kac algebras is necessary for the materialization of duality. One, called the Abelian Kac algebra, has the L^∞-functions for elements. The other, the symmetric Kac algebra, includes the left-regular representations. The problem lies in the fact that the symmetric Kac algebra for the Heisenberg group H_3 is generated by (linear!) left-regular representations, while the Weyl kernels are irreducible projective operators. This is reviewed in Section 4. To go from the H_3-Kac duality to the desired projective algebras, two steps are involved: projection and decomposition into irreducibles, in this order or its inverse. The extensions so obtained are far from trivial. Kac algebras are, to begin with, Hopf–von Neumann algebras, and the necessary extensions involve generalizations of some of the current concepts on Hopf algebras. Though most of the axioms remain unchanged, some of the usual requirements valid for linear representations must be extended to their projective counterparts. This generalization to projective Kac algebras is presented in Section 5. It leads to the general notion of projective Kac algebra. An extended duality comes out, a projective Kac duality leading to a projective Fourier duality. It is necessary to introduce such notions as projective coinvolution and coprojective coinvolution as well as to extend the usual axioms concerning anti-(co)automorphisms. In Section 6 the Weyl–Wigner correspondence is recast into the Fourier duality language. Weyl's formula shows up as an irreducible component of the duality mapping between the previously obtained projective algebras. A tentative prescription for quantization on general phase spaces is sketched in the final considerations.

2. CLASSICAL AND QUANTUM MECHANICS IN PHASE SPACE

The classical picture of mechanics, as is well known, can be described geometrically in terms of the symplectic structure of the phase space (Arnold, 1978). The simplest symplectic manifold is just $\mathbf{R}^2$, the most usual arena of classical dynamical systems with one degree of freedom. Acting by symplectomorphisms on this phase space, with its trivial symplectic structure $\omega = dp \wedge dq$, is the Abelian group—also denoted $\mathbf{R}^2$—of two-dimensional translations. To classical Hamiltonian dynamical systems on $\mathbf{R}^2$ associated to Hamiltonian functions H there correspond symplectic Hamiltonian vector fields X_H satisfying the Hamilton equations

$$i_{X_H}\omega = -dH \tag{1}$$

The nondegeneracy of the symplectic form implies a local isomorphism between vector fields and 1-forms [see (1)] and a homomorphism between vector fields and C^∞-functions. Vector fields constitute a Lie algebra by the Lie bracket, whose isomorphic image on the space of functions is the Poisson bracket

$$\{f, g\} = -\omega(X_f, X_g)$$

According to Dirac (1958), in order to quantize on such a space, we must be sure that there exists a faithful correspondence between this Poisson algebra and an operatorial algebra. The closest operator algebra we have at hand is the Lie algebra of the group acting on the phase space by symplectomorphisms. In the Euclidean case, since the translation group is Abelian, we must central-extend it to the Heisenberg group in order to have the isomorphism of the group and the Poisson Lie algebras. Such an isomorphism allows the construction of a faithful quantization map on this phase space (Isham, 1984).

From the point of view of harmonic analysis, the Pontryagin duality for the Abelian group $\mathbf{R}^2$ ensures that the Fourier transform and its inverse constitute an isomorphism between the Abelian convolution algebra $L^1(\mathbf{R}^2)$ and also the Abelian algebra $L^\infty(\mathbf{R}^2)$ of essentially bounded functions with pointwise product, both contained in $C^\infty(\mathbf{R}^2)$. The Fourier transform of an L^1-function f is the L^∞-function

$$[\mathcal{F}f](x, y) = \tilde{f}(x, y) = \frac{1}{2\pi}\int_{\mathbf{R}^2} dq\, dp\, f(p, q)e^{-i(yq+xp)} \tag{2}$$

where the kernel $e^{i(yq+xp)} \equiv \chi_{(x,y)}(q, p)$ is a character (one-dimensional irreducible representation) of the group $\mathbb{R}^2$. Characters satisfy the orthogonality-completeness relation

$$\int_{\mathbb{R}^2} dx\, dy\, \chi_{(x,y)}(q, p)\overline{\chi_{(x,y)}(q', p')} = (2\pi)^2\delta(q - q')\delta(p - p')$$

which can be used to invert (2) and write the inverse Fourier transform as

$$f(p, q) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dx\, dy\, \tilde{f}(x, y)e^{i(yq+xp)} \tag{3}$$

From this point of view, the correct way to regard these formulas is to see the first as the Fourier transform from $L^1(\mathbb{R}^2)$ into $L^\infty(\hat{\mathbb{R}}^2)$, which is the same as $L^\infty(\mathbb{R}^2)$, since this group is self-dual, $\hat{\mathbb{R}}^2 = \mathbb{R}^2$, and the second as the transform from $L^1(\hat{\mathbb{R}}^2)$ into $L^\infty(\mathbb{R}^2)$. This is so because the Fourier transform is defined as a mapping between the L^1-space of an Abelian group into the L^∞-space of its dual, the space of characters, which is also a group. Because $\mathbb{R}^2$ is self-dual, the Fourier transform turns out to be an algebra isomorphism, mapping convolution to pointwise product (Reiter, 1968). This enables us to regard formula (3) as the inversion of (2), or as the Fourier transform for $\hat{R}^2$. Thus, from the point of view of harmonic analysis, this group is highly "degenerate." Extending the domain of the transform $\mathcal{F}$ from L^1 to the space $\mathcal{S}'(\mathbb{R}^2)$ of tempered distributions on $\mathbb{R}^2$, $\mathcal{F}$: $\mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$, turns out to be a topological isomorphism (Sugiura, 1990; Choquet-Bruhat *et al.*, 1982). The same happens when $\mathcal{F}$ is restricted to the space $\mathcal{S}(\mathbb{R}^2)$ of rapidly decreasing functions, confirming the degeneracy alluded to.

To go from this classical approach to a quantum picture, Weyl (1931) proposed to modify the Fourier transform formula by changing its scalar kernel into an operatorial kernel. He wrote

$$\hat{f}_\hbar = \int_{\mathbb{R}^2} dq\, dp\, f(q, p)e^{-(i/\hbar)(p\hat{q}+q\hat{p})} \tag{4}$$

instead of $\tilde{f}$, where $\hat{q}$, $\hat{p}$ are the usual coordinate and momentum operators of Euclidean quantum mechanics. These operators satisfy the Heisenberg commutation relation $[\hat{q}, \hat{p}] = i\hbar$. By the Glauber identity, the operatorial kernel

$$S'_\hbar(x, y) \equiv e^{-(i/\hbar)(y\hat{q}+x\hat{p})}$$

can also be written as

$$S_\hbar(x, y) = e^{(i/2\hbar)xy}U(y)V(x)$$

in terms of the Weyl operators

$$U(y) = e^{-(i/\hbar)y\hat{q}} \qquad \text{and} \qquad V(x) = e^{-(i/\hbar)x\hat{p}}$$

These satisfy the Weyl commutation relation (we use the representation theory convention, by which the leftmost operator acts first)

$$U(y)V(x) = e^{-(i/\hbar)xy}V(x)U(y)$$

while for $S_\hbar$

$$S_\hbar(x, y)S_\hbar(x', y') = e^{(i/2\hbar)(xy'-yx')}S_\hbar(x + x', y + y') \tag{5}$$

These are consequently not linear, but projective operators. They realize an operatorial representation of R^2, otherwise impossible for such an Abelian group. Summing up, as a realization of the classical–quantum correspondence principle, Weyl proposed with (4) to consider the passage from the usual scalar Fourier transform to an operatorial one written in terms of projective representations. The inverse way (Wigner, 1932), leading from the quantum to the classical picture, involves an *integration* on operator space (taking of the trace):

$$f(q, p) = \mathrm{Tr}[S_\hbar^\dagger(q, p)\hat{f}_\hbar] \tag{6}$$

Since the projective operator product (5) carries a natural twisting, and since formulas (4), (6) ought to represent an algebra isomorphism, the corresponding convolution in $L^1(R^2)$ also gets twisted:

$$\begin{aligned}\hat{f}_\hbar \cdot \hat{g}_\hbar &= \int_{R^2\times R^2} dx\, dy\, dx'\, dy'\, f(x, y)g(x', y') \\ &\quad \times e^{(i/2\hbar)(xy'-yx')}S_\hbar(x + x', y + y') \\ &= \int_{R^2} dx''\, dy''\, (f \circledast g)(x'', y'')S_\hbar(x'', y'')\end{aligned}$$

where

$$(f \circledast g)(x'', y'') = \int_{R^2} dx\, dy\, e^{(i/2\hbar)(xy''-yx'')}f(x, y)g(x'' - x, y'' - y)$$

Formulas (4) and (6) provide a two-way correspondence between the classical (L^1-functions) and the quantum pictures. The further correspondence to L^∞-functions is provided by the Fourier transform. The Fourier transform of a twisted convolution of two functions gives rise to the twisted (noncommutative) product of their Fourier transforms, which characterizes a deformation of the Abelian algebra of the pointwise product. This two-way classical–quantum procedure is the *Weyl–Wigner correspondence prescription.*

Projective representations of a group are generally obtained from the linear representations of its central extension (Bargmann, 1954), in our case the Heisenberg group. Harmonic analysis on general locally compact groups like the Heisenberg group is not as trivial as that on the Abelian ones. Because such groups have infinite-dimensional irreducible representations, finite algebras do not have enough structure to host a duality. We must deal with semifinite von Neumann algebras endowed with additional structure, the Kac algebras. From the relation between the irreducible representations of the Heisenberg group and the projective operators appearing in the Weyl–Wigner formalism, we can relate harmonic analysis on the Heisenberg group to a projective harmonic analysis on R^2 and "explain" the origin of the Weyl–Wigner formulas. This will be done in the final sections, after we have established some facts on the Heisenberg group in Section 3 and reviewed the duality theory for it in terms of Kac algebras in Section 4.

3. THE HEISENBERG GROUP

In this work the three-dimensional Heisenberg group H_3 is regarded as the central extension of the two-dimensional Abelian group of translations on the plane by the torus T. We shall use the notation $(x, \alpha) = (x_1, x_2, e^{i\theta})$, $x_1, x_2 \in \mathrm{R}$, $\theta \in \mathrm{R}/2\pi$, to denote the elements and coordinates of H_3. As is well known, the second cohomology space $H^2(\mathrm{R}^2, \mathrm{R}/2\pi)$ of cocycles from R^2 to R (mod 2π) is not trivial (Tuynman and Wiegerinck, 1987). Since 2-cocycles classify central extensions, inequivalent 2-cocycles give rise to inequivalent central extensions. Thus, for a chosen cocycle $\Omega \in H^2(\mathrm{R}^2, \mathrm{R}/2\pi)$, e.g.,

$$\Omega(x, y) = \tfrac{1}{2}(x_1 y_2 - y_1 x_2) \tag{7}$$

the product on $H_3 = \mathrm{R}^2 \times \mathrm{T}$ is given by

$$(x, \alpha)(y, \beta) = (x + y, \alpha\beta e^{i\Omega(x,y)})$$

where associativity is ensured by the closeness of Ω in H^2, namely

$$\delta\Omega(x, y, z) = \Omega(y, z) - \Omega(x + y, z) + \Omega(x, y + z) - \Omega(x, y) = 0$$

The identity in H_3 is $(0, 1)$ and the inverse element of (x, α) is $(-x, \alpha^{-1})$. The following useful properties of Ω are obvious from (7): $\Omega(x, 0) = 0$, $\Omega(x, y) = -\Omega(y, x)$, $\Omega(-x, y) = -\Omega(x, y)$.

The irreducible linear representations of H_3 can be obtained by Mackey's induced representation method (Mackey, 1987; Taylor, 1986). Their division into inequivalent classes is given by the Stone–von Neumann theorem, which also provides the unitary dual space of this group. These representations are divided into infinite-dimensional and one-dimensional ones in the dual $\hat{H}_3 = (\mathbb{Z} - \{0\}) \cup \mathbb{R}^2$, according to

$$T_\nu(x, \alpha) = e^{i\nu\theta} e^{(i/2)\nu x_1 x_2} e^{-i\nu x_2 \hat{q}} e^{-ix_1 \hat{p}}, \qquad \nu \in \mathbb{Z} - \{0\} \tag{8a}$$

$$T_{ab}(x, \alpha) = e^{iax_2} e^{ibx_1}, \qquad (a, b) \in \mathbb{R}^2 \tag{8b}$$

where the self-adjoint operators $\hat{q}$, $\hat{p}$ act on $L^2(\mathbb{R})$ by

$$\hat{q}\psi(q) = q\psi(q)$$

$$\hat{p}\psi(q) = -i\partial_q \psi(q)$$

We recall that the commutation relation

$$[\hat{q}, \hat{p}] = i$$

is a realization of the Lie algebra of H_3 on that Hilbert space, which is also isomorphic to the Poisson algebra generated by the coordinates q, p plus the constant function 1 on the Euclidean symplectic manifold $\mathbb{R}^2$.

4. FOURIER DUALITY FOR THE HEISENBERG GROUP

Our objective is to describe the Weyl–Wigner correspondence in terms of projective Fourier duality, that is, to find a connection between Kac duality and the algebra generated by irreducible projective operators. Since the projective representations of $\mathbb{R}^2$ are obtained from the linear representations of H_3, we should start from the well-established Kac algebra duality for this group. This is reviewed in this section. As already said, duality requires a pair of Kac algebras: the Abelian Kac algebra, formed with the L^∞-functions, and the symmetric Kac algebra, including the left-regular representations.

4.1. The Symmetric Kac Algebra of H_3

Let us begin by introducing the symmetric Kac algebra of H_3, $\mathbb{K}^s(H_3)$, which is built on the von Neumann algebra $\mathcal{M}(H_3)$ generated by the left-regular representation operators of the group. For details on Kac algebras, see Enock and Schwartz (1992) and the quick review in the first sections of Vaĭnermann (1988), or Aldrovandi and Saeger (1996). First recall that the left-regular representation L acts on the Hilbert space $L^2(H_3)$ of square-integrable functions on the Heisenberg group by

$$[L(x, \alpha)f](y, \beta) = f((x, \alpha)^{-1}(y, \beta)) \tag{9}$$

The scalar product in this space is given by

$$(f|g)_{L^2(H_3)} = \int_{H_3} dx\, d\alpha\, f(x, \alpha)\overline{g(x, \alpha)}$$

and the norm by $\|f\|_2^2 = (f|f)_{L^2(H_3)}$, where $dx\, d\alpha$ is the left- and right-invariant measure on H_3, which is unimodular. In this section the spaces $L^p(H_3)$, $p = 1, 2, \infty$, will be denoted simply by L^p.

$\mathcal{M}(H_3)$ is a subalgebra of $\mathcal{B}(L^2)$, the Banach algebra of all bounded operators on the Hilbert space L^2. This means that the product on $\mathcal{M}(H_3)$ is associative, there exists a unit $I = L(0, 1)$ (the identity operator), and also an involution (taking of the dagger) such that $I^\dagger = I$. The norm is defined by $\|T\| = \sup\{\|T\psi\|_2, \|\psi\|_2 = 1\}$, under which $\|T^\dagger\| = \|T\|$ [$\mathcal{M}(H_3)$ is an involutive Banach algebra] and $\|T^\dagger T\| = \|T\|^2$ (it is a C^*-algebra). There is also a family of seminorms defined by $\|T\|_w = |(\psi | T\phi)_{L^2}|$, $\psi, \phi \in L^2$, whose open balls define the *weak topology*. $\mathcal{M}(H_3)$ is closed in this topology.

The elements of $K^s(H_3)$ are written in terms of the generators as

$$\hat{f} = \int_{H_3} dx\, d\alpha\, f(x, \alpha) L(x, \alpha) \tag{10}$$

where the coefficients f are functions of compact support, whose algebra $C(H_3)$ is dense in the convolution Banach algebra L^1. The product can be written in terms of the convolution of the coefficients:

$$\begin{aligned}\hat{f} \cdot \hat{g} &= \int_{H_3 \times H_3} dx\, d\alpha\, dy\, d\beta\, f(x, \alpha) g(y, \beta) L((x, \alpha)(y, \beta)) \\ &= \int_{H_3} dz\, d\gamma\, (f * g)(z, \gamma) L(z, \gamma)\end{aligned}$$

where the convolution on $C(H_3)$ is written

$$(f * g)(z, \gamma) = \int_{H_3} dx\, d\alpha\, f(x, \alpha) g((x, \alpha)^{-1}(z, \gamma))$$

Formula (10) can also be regarded as expressing the left-regular representation of L^1 induced by the left-regular representation of the group, and as such is denoted $L(f)$.

With such elements, $K^s(H_3)$ has a structure given by the following operations:

- A *product* given by the group multiplication,

$$L(x, \alpha) L(y, \beta) = L(x + y, \alpha\beta e^{i\Omega(x,y)}) \tag{11}$$

- A symmetric (wherefrom the name of this algebra) *coproduct*,

$$\hat{\Delta} L(x, \alpha) = L(x, \alpha) \otimes L(x, \alpha) \tag{12}$$

- A *coinvolution*,

$$\hat{\kappa}(L(x, \alpha)) = L^{\dagger}(x, \alpha) = L((x, \alpha)^{-1}) \tag{13}$$

- A normal, faithful, and semifinite (n.f.s) Haar *trace*,

$$\hat{\varphi}(T) = \begin{cases} \|f\|_2^2 & \text{if} \quad T = \hat{f}^{\dagger} \cdot \hat{f} \\ +\infty & \text{otherwise} \end{cases} \qquad T \in \mathcal{M}(H_3)^+ \tag{14}$$

Normal, faithful, and semifinite mean, respectively, that: $\hat{\varphi}(T)$ is the upper bound of the sequence $\{\hat{\varphi}(T_i)\}$ if $T \in \mathcal{M}(H_3)^+$ is the upper bound of the sequence $\{T_i\}$; $\hat{\varphi}(T) = 0$ implies $T = 0$, $T \in \mathcal{M}(H_3)^+$; the algebra $span\{T \in \mathcal{M}(H_3)^+ \mid \hat{\varphi}(T) < \infty\}$ is σ-weakly dense in $\mathcal{M}(H_3)$. The σ-weak topology is defined by the open balls of the seminorms

$$\|F\|_{\sigma,\psi_i,\phi_i} = \sum_i |(\phi_i | F\psi_i)|$$

where

$$\sum_i \|\phi_i\|^2 < \infty, \qquad \sum_i \|\psi_i\|^2 < \infty$$

$\mathcal{M}(H_3)^+$ is the set of positive elements of $\mathcal{M}(H_3)$, that is, the set of operators with positive spectrum.

Equation (14) is coherent with

$$\hat{\varphi}(\hat{f}) = f(0, 1)$$

for in this case $\hat{\varphi}(\hat{f}^{\dagger} \cdot \hat{f}) = (f^* * f)(0, 1) = \|f\|_2^2$, where the asterisk denotes the involution on L^1 given by

$$f^*(x, \alpha) = \overline{f((x, \alpha)^{-1})} \tag{15}$$

The coproduct has a canonical implementation on $\mathcal{M}(H_3)$ in terms of a unitary operator $\hat{W} \in \mathcal{B}(L^2) \otimes L^{\infty}$,

$$\hat{\Delta}L(x, \alpha) = \hat{W}(I \otimes L(x, \alpha))\hat{W}^* \tag{16}$$

This *fundamental operator* is unique and is fixed by

$$[\hat{W}F](x, \alpha; y, \beta) = F((y, \beta)^{-1}(x, \alpha); (y, \beta)) \tag{17}$$

where $F \in C(H_3 \times H_3)$. Its adjoint $\hat{W}^*$ is given by

$$[\hat{W}^*F](x, \alpha; y, \beta) = F((y, \beta)(x, \alpha); (y, \beta))$$

The importance of $\hat{W}$ and its dual $W = \sigma \circ \hat{W}^* \circ \sigma$ lies in that they *generate the Kac duality*, in the sense that they are the generators of the representations linking $\mathrm{K}^s(H_3)$ and its dual. As a consequence of that and of (16), they satisfy the *pentagonal relation*

$$(I \otimes \hat{W})(\sigma \otimes I)(I \otimes \hat{W})(\sigma \otimes I)(\hat{W} \otimes I) = (\hat{W} \otimes I)(I \otimes \hat{W})$$

In the same way, the coinvolution has a canonical implementation in terms of the antilinear isometry J: $L^2 \to L^2$ by

$$\hat{\kappa}(L(x, \alpha)) = JL^{\dagger}(x, \alpha)J$$

The latter is given on $C(H_3)$ by $[Jf](x, \alpha) = \overline{f(x, \alpha)}$, and in the case of an unimodular group like H_3, it also implements the involution in the (pre)-dual algebra.

4.2. The Abelian Kac Algebra of H_3

In order to be a Kac algebra, $\mathrm{K}^s(H_3) = (\mathcal{M}(H_3), \hat{\Delta}, \hat{\kappa}, \hat{\varphi})$ must satisfy a certain set of axioms. These will be presented later in Section 5.1.1 on the projection process. We only anticipate that $\mathrm{K}^s(H_3)$, as introduced above, does satisfy them. For the time being we are interested in duality for H_3. The dual of $\mathrm{K}^s(H_3)$ is obtained as the image of the *Fourier representation* $\hat{\lambda}$ of the predual of $\mathcal{M}(H_3)$. The predual $\mathcal{M}(H_3)_*$, which is isomorphic to the *Fourier algebra* $A(H_3)$ of H_3, is the space of all σ-weakly continuous linear functionals on $\mathcal{M}(H_3)$. The representative elements of $\mathcal{M}(H_3)$ are linear forms $\hat{\omega}_{fg}$ on L^2, in terms of which the corresponding functions in $A(H_3)$ are defined by

$$\hat{\omega}_{fg}(x, \alpha) \equiv \langle L^{\dagger}(x, \alpha), \hat{\omega}_{fg}\rangle = (f * \check{g})(x, \alpha) \in A(H_3), \qquad f, g \in L^2 \tag{18}$$

where $\langle L(x, \alpha), \hat{\omega}_{fg}\rangle \equiv (L(x, \alpha)f | g)_{L^2}$ by definition of $\hat{\omega}_{fg}$, and

$$\check{g}(x, \alpha) = \overline{g((x, \alpha)^{-1})}$$

Notice that, by applying the Cauchy–Schwartz inequality to (18), we find that this function has an upper bound, that is, $|\hat{\omega}_{fg}(x, \alpha)| \leq \|f\|_2 \|g\|_2 < \infty$, and consequently $A(H_3) \subset L^{\infty}$. The product in the predual $A(H_3)$ is obtained by duality from the coproduct in $\mathrm{K}^s(H_3)$,

$$\langle L^{\dagger}(x, \alpha), \hat{\omega}_{fg} \cdot \hat{\omega}_{hl}\rangle = \langle \hat{\Delta} L^{\dagger}(x, \alpha), \hat{\omega}_{fg} \otimes \hat{\omega}_{hl}\rangle \tag{19}$$

and, as follows trivially from (12), is the Abelian pointwise product. The involution ^{o} in $A(H_3)$ also follows by duality from

$$\langle L^{\dagger}(x, \alpha), \hat{\omega}_{fg}^{o}\rangle = \overline{\langle \kappa(L^{\dagger}(x, \alpha))^{\dagger}, \hat{\omega}_{fg}\rangle} \tag{20}$$

and is simply the complex conjugation implemented by J.

To find the Fourier representation $\hat{\lambda}$, defined by

$$[\hat{\lambda}(\hat{\omega})f](x, \alpha) = [(\hat{\omega} \circ \hat{\kappa} \otimes id)(\hat{\Delta}\hat{f})]_{\hat{\varphi}}(x, \alpha), \qquad f \in L^2 \tag{21}$$

we may use the formula

$$(\hat{W}(f \otimes g) \mid h \otimes l)_{L^2 \otimes L^2} = (g \mid \hat{\lambda}(\hat{\omega}_{hf})l)_{L^2}, \qquad f, g, h, l \in L^2 \tag{22}$$

which relates $\hat{\lambda}$ to the dual of its generator W. Computing the double scalar product in (22), taking into account (17), and identifying $\hat{\omega}_{hf}$ from (18), we get $\hat{\lambda} = id$. This means that the Kac algebra dual to $\mathbb{K}^s(H_3)$ is built on the von Neumann algebra L^∞ of measurable and essentially bounded functions on the Heisenberg group. The coproduct, coinvolution, and trace thus obtained, together with the pointwise product and the involution, satisfy the Kac algebra axioms. This Abelian Kac algebra $\mathbb{K}^s(H_3)$ on L^∞ is then defined by the following structure:

$$[f \cdot g](x, \alpha) = f(x, \alpha) g(x, \alpha) \tag{23a}$$

$$\mathbf{1} = 1 \qquad \text{such that} \quad 1(x, \alpha) = 1 \quad \forall (x, \alpha) \tag{23b}$$

$$\Delta(f)((x, \alpha) \otimes (y\ \beta)) = f((x, \alpha)(y, \beta)) \tag{23c}$$

$$\kappa(f)(x, \alpha) = f((x, \alpha)^{-1}) \tag{23d}$$

$$\varphi(f) = \int_{H_3} dx\, d\alpha\, f(x, \alpha), \qquad f \in L^{\infty +} \tag{23e}$$

The positive elements are the positive-definite functions in L^∞. This von Neumann algebra is also a subalgebra of $\mathcal{B}(L^2)$, which acts on L^2 by pointwise multiplication. Its norm is given by $\|f\|_\infty = ess.sup. |f(x)|$, which is the smallest number C ($0 \leq C < \infty$) such that $|f(x)| \leq C$ locally almost everywhere (Reiter, 1968). The predual of L^∞ is just L^1, the convolution algebra with involution given by (15). Its structure is also obtainable by duality relations similar to (19) and (20), but now between L^∞ and L^1. The fundamental operator for this algebra is W, which implements Δ and is given by

$$[WF](x, \alpha; y, \beta) = F((x, \alpha); (x, \alpha)(y, \beta))$$

while the dual $\hat{J}$ of J is given by $[\hat{J}f](x, \alpha) = \overline{f(-x, \alpha^{-1})}$, in terms of which we have $\kappa(f) = \hat{J} f \hat{J}$ on L^2.

The duality $\mathbb{K}^s(H_3) - \mathbb{K}^a(H_3)$ for the Heisenberg group will be complete when $L^\infty_* = L^1$ is represented in $\mathcal{M}(H_3)$. This is carried out by the Fourier representation λ, dual of $\hat{\lambda}$, which is just the regular representation of L^1 restricted to act on $L^\infty \cap L^2$. Another way to see that, and in fact to deduce it, is to use the dual of formula (22),

$$(W(f \otimes g) \mid h \otimes l)_{L^2 \otimes L^2} = (g \mid \lambda(\omega_{hf})l)_{L^2}, \qquad f, g, h, l \in L^2 \tag{24}$$

where $\omega_{fg} \in L^1$ is defined by $\langle h, \omega_{fg} \rangle = (hf \mid g) \therefore \omega_{fg} = f\bar{g}$. We obtain

$$\lambda(f) = \int_{H_3} dx\, d\alpha\, f(x, \alpha)L(x, \alpha), \qquad f \in L^1$$

whose image is just $\mathcal{M}(H_3)$. Recall that formula (24) is a consequence of the dual $\hat{W} = \sigma \circ W^* \circ \sigma$ being the generator of λ. To see that and to understand what the word *generator* really means, define $\phi \in L^2(H_3, L^2)$ by $[\phi(y, \beta)](x, \alpha) = F(x, \alpha; y, \beta)$, where $F \in L^2 \otimes L^2$. These spaces are isomorphic. Recall also that, as a representation, the operator L is a bounded map between H_3 and $\mathcal{B}(L^2)$. Then for (y, β) fixed, $L(y, \beta) \in \mathcal{B}(L^2)$, $\phi(y, \beta) \in L^2$, and we have

$$\begin{aligned}[L(y, \beta)\phi(y, \beta)](x, \alpha) &= [\phi(y, \beta)]((y, \beta)^{-1}(x, \alpha)) \\ &= F((y, \beta)^{-1}(x, \alpha); (y, \beta))\end{aligned}$$

which is just $[\hat{W}F](x, \alpha; y, \beta)$ as given in (17). That is, $L\colon H_3 \to \mathcal{B}(L^2)$, which induces (generates) λ, can be seen as the operator $\hat{W} \in \mathcal{B}(L^2) \otimes L^\infty$. This is put in compact form as $\lambda(f) = (id \otimes f)(\hat{W})$.

As a final remark regarding such Kac algebras, notice that both $\mathbb{K}^a(H_3)$ and $\mathbb{K}^s(H_3)$ are represented on $L^2(H_3)$ by the Gelfand–Naimark–Segal (GNS) construction. The first is represented by the inclusion of L^∞ and the latter by the inclusion of the L^1-coefficients.

5. PROJECTIVE KAC ALGEBRAS

In this section we project the symmetric and Abelian Kac algebras of H_3 into algebras related to the projective representations of $\mathbb{R}^2$ and obtain a projective duality extension. It is worthwhile to spend some time on the definition of the projective representations, since they are crucial for the projection process.

We start from Bargmann's (1954) method to obtain projective representations of a group from the linear representations of its central extensions. The representations of the central extension giving rise to projective representations are those reducing to the identity when restricted to the central subgroup. In our case we have the left-regular representations of H_3, which act on $L^2(H_3)$ by (9). As before, we will concentrate on the central extension defined by the cocycle Ω introduced in (7). It is clear from (9) that the restriction of L to $\mathbb{T}$ is not the identity representation, so that we must make L act on another space, suitable to our purposes. By Mackey's induced representation method (Mackey, 1987), the left-regular representation can be regarded as induced by the identity representation of the subgroup $\{e\}$. If in the induction process we change any other subgroup for $\{e\}$, the resulting representation is called *quasiregular* (Barut and Raczka, 1977). Thus, since $\mathbb{T}$ is central, its behavior is equivalent to that of $\{e\}$, which enables us to interpret the representations induced by the identity representation of T as the regular representations acting on another space. By that method, $L(x, \alpha)$ should act on a Hilbert space isomorphic to $L^2(R^2)$, which we call here $H(H_3)$. Its elements are square-integrable functions when restricted to R^2 and furthermore satisfy $f((x, \alpha)(0, \beta)) = \beta^{-1}f(x, \alpha)$. Since $(x, \alpha) = (x, 1)(0, \alpha) = (0, \alpha)(x, 1)$, we have the decomposition

$$f(x, \alpha) = \alpha^{-1}f(x, 1) \equiv \alpha^{-1}f(x) \tag{25}$$

where the same notation f for functions on H_3 and on R^2 is used. By this natural *projection* of $L^2(H_3)$ into $L^2(R^2)$, (9) can be rewritten as

$$[L(x, \alpha)f](y) = \alpha e^{-i\Omega(-x,y)}f(y - x)$$

which does reduce to the identity when restricted to T, namely $[L(0, \alpha)f](y) = \alpha f(y)$. The respective projective representation of R^2 is then defined on $L^2(R^2)$ by

$$[L_\Omega(x)f](y) \equiv [L(x, 1)f](y) = e^{i\Omega(x,y)}f(y - x) \tag{26}$$

From what has been said above we can also write the decomposition of $L(x, \alpha)$ as

$$L(x, \alpha) = \alpha L_\Omega(x) \tag{27}$$

As a consequence, the R^2 operation (sum) is now represented by

$$L_\Omega(x)L_\Omega(y) = e^{i\Omega(x,y)}L_\Omega(x + y) \tag{28}$$

which characterizes a projective representation.

5.1. Projective Kac Algebras of the Translation Group

We proceed now to project $K^s(H_3)$ and $K^a(H_3)$ according to the decomposition (27). Let us begin by observing that, although Ω is not trivial in $H^2(R^2, R/2\pi)$, it is exact in another group cohomology. If a complex of *gaugefied* (that is, point-dependent) k-cochains $R \times (R^2)^{\otimes k} \to R$ with a derivative δ' is considered, then there exists a 1-cochain Θ such that $\Omega = \delta'\Theta$, or

$$\Omega(x, y) = \delta'\Theta(x, y) = \Theta(y\cdot q; x) - \Theta(q; x + y) + \Theta(q; y) \tag{29}$$

where $y\cdot q \equiv q + y_1$ is an action of R^2 on R (Aldrovandi and Galetti, 1990). One such Θ is given explicitly by

$$\Theta(q; x) = -\tfrac{1}{2}[(2q + x_1)x_2] \tag{30}$$

and satisfies $\Theta(q; 0) = 0, \Theta(q; -x) = -\Theta(x^{-1}\cdot q; x), \Theta(q; x) = -\Theta(x\cdot q; -x)$, $\forall q \in R, x \in R^2$. By direct calculation one also obtains the interesting property

$$\Theta(q; x) - \Theta(y\cdot q; x) = \Theta(q; -x) - \Theta(y^{-1}\cdot q; -x) = y_1x_2 \tag{31}$$

This kind of 1-cochain appears naturally in representation theory. For example, representations (8a) on $L^2(\mathrm{R})$ can be written in terms of Θ as follows (Varadarajan, 1970):

$$[T_\nu(x, \alpha)f](q) = e^{i\nu\theta} e^{i\nu\Theta(x^{-1}\cdot q; x)} f(x^{-1}\cdot q), \qquad \nu \in \mathbb{Z} - \{0\} \qquad (32)$$

With these remarks in mind, we reinterpret the decomposition formula (27), and consider that the central element $\alpha = e^{i\theta} \in \mathbb{T}$ appears in the projection in the form

$$L(x, \alpha) \mapsto e^{i\Theta(q; -x)} L_\Omega(x) \qquad (33)$$

This means that we will take $\theta = \Theta$ and regard $\Theta = \Theta(q; -x)$ as a gaugefied 1-cochain as defined above. This means that it depends on the projected point $x \in \mathrm{R}^2$, which is its partner in the H_3 coordinates, and, furthermore, it is gaugefied—it depends on the point $q \in \mathrm{R}$ where the irreducible representations (32) [irreducible components of $L(x, \alpha)$] act. Despite the local character of Θ as regards the first two slots (x) of the H_3 coordinates, in the projection formula (33) Θ will as a whole account for the third slot irrespective of the details of its content. For example, $L(x, \alpha\beta)$ and $L(x, \alpha^{-1})$ will also be projected into the right-hand side of (33), but $L(-x, \alpha)$ will be projected to $e^{i\Theta(q;x)} L_\Omega(-x)$.

5.1.1. The Projection of $\mathrm{K}^s(H_3)$

Let us project $\mathrm{K}^s(H_3)$ according to the map (33), in order to find the structure of the space generated by the operators $L_\Omega(x)$, $x \in \mathrm{R}^2$. This will be done in two steps: (i) the projection of operations like norm, involution, product, etc., and (ii) the verification of the Kac algebra axioms for these projected operations. Here the Kac algebra axioms will just play the role of guiding axioms, since the resulting algebra is not exactly a Kac one.

Let us begin by projecting the norm and the involution. Since $e^{i\Theta(q;-x)}$ is a complex number, from (33) we have simply

$$\|L(x, \alpha)\| \mapsto \|L_\Omega(x)\| = sup\{\|L_\Omega(x)\psi\|_2, \|\psi\|_2 = 1 \text{ in } L^2(\mathrm{R}^2)\}$$

The dagger in $\mathcal{M}(H_3)$ is projected to

$$L^\dagger(x, \alpha) \mapsto (e^{i\Theta(q;-x)})^* L_\Omega^\dagger(x) \qquad (34)$$

where, since these representations are unitary, $L_\Omega^\dagger(x) = L_\Omega(-x)$. The involution * on the phase factor is not simply complex conjugation, but involves also the action of R^2 on R. It becomes fixed if we recall that $L^\dagger(x, \alpha) = L(-x, \alpha^{-1}) \mapsto e^{i\Theta(q;x)} L_\Omega(-x)$, and compare with (34), which gives

$$(e^{i\Theta(q;-x)})^* = e^{i\Theta(q;x)} \qquad (35)$$

It is easy to verify that the projected $\|\cdot\|$ and $\dagger$ satisfy all the usual norm and involution axioms [see them in Bratteli and Robinson (1987)].

The product in $K^s(H_3)$ will also be projected according to (33). Care must be taken when dealing with such products of operators. Since an operator at the right *feels* the action of that at the left on R, its phase factor turns out to be modified. From (11) and the above we have

$$e^{i\Theta(q;-x)}L_\Omega(x)e^{i\Theta(x^{-1}\cdot q;-y)}L_\Omega(y) = e^{i\Theta(q;-x-y)}L_\Omega(x + y) \tag{36}$$

which only gives (28) [by (29) and $\Omega(-y, -x) = -\Omega(x, y)$] if $\Omega = \delta'\Theta$. This implies that Θ can be given by (30). The first condition imposed on the product is associativity, which is satisfied due to the closedness of the cocycle Ω. The second, $L_\Omega(x)\mathbf{1} = \mathbf{1}L_\Omega(x)$ $\forall x$, where $\mathbf{1} = L_\Omega(0)$, is true for (28) because $\Omega(\cdot, 0) = 0$.

Up to this point we have a unital, involutive, and normed algebra with product given by (28). It is also a subalgebra of $\mathcal{B}(L^2(\mathrm{R}^2))$, and is certainly closed in the weak topology defined on it. This can be seen by comparison with the von Neumann algebra generated by the left-regular operators $L(x)$ of R^2. The only difference between the action of $L_\Omega(x)$ and the action of $L(x)$ on $L^2(\mathrm{R}^2)$ is a phase factor [see (26)], which does not affect the closedness property in the weak topology (see Section 4), for example. The conclusion is that the algebra generated by $L_\Omega(x)$, $x \in \mathrm{R}^2$, is a von Neumann algebra. It will be denoted $\mathcal{M}^\Omega(\mathrm{R}^2)$.

Going further, by (33) we project the coproduct to

$$\hat{\Delta}L(x, \alpha) \mapsto e^{i\Theta(q;-x)}\Delta^\Omega L_\Omega(x)$$

where $\hat{\Delta}$ is not supposed to act on the central element $e^{i\Theta}$. Taking this in the expression for the coproduct of $L(x, \alpha)$ and considering again the projection formula, we get

$$\Delta^\Omega L_\Omega(x) = e^{i\Theta(q;-x)}L_\Omega(x) \otimes L_\Omega(x) \tag{37}$$

Notice that it remains symmetric, that is, $\sigma \circ \Delta^\Omega = \Delta^\Omega$, where $\sigma(T \otimes T') = T' \otimes T$, as was $\hat{\Delta}$ on $K^s(H_3)$. The first axiom the coproduct must satisfy is $\Delta^\Omega\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$, which is trivial, since $\Theta(q; 0) = 0$ for all $q \in \mathrm{R}$. The next is *coassociativity*, which means

$$(\Delta^\Omega \otimes id) \circ \Delta^\Omega = (id \otimes \Delta^\Omega) \circ \Delta^\Omega$$

It is also trivial since the same phase factor occurs twice on both sides of this equation when it is applied to $L_\Omega(x)$. Finally, Δ^Ω should be a homomorphism from $\mathcal{M}^\Omega(\mathrm{R}^2)$ to $\mathcal{M}^\Omega(\mathrm{R}^2) \otimes \mathcal{M}^\Omega(\mathrm{R}^2)$, which means that

$$\Delta^\Omega(L_\Omega(x)L_\Omega(y)) = \Delta^\Omega(L_\Omega(x))\Delta^\Omega(L_\Omega(y)) \tag{38}$$

The left-hand side of equation (38) yields

$$e^{i[\Omega(x,y)+\Theta(q;-x-y)]}L_\Omega(x+y)\otimes L_\Omega(x+y)$$

while its right-hand side gives

$$e^{i[\Theta(q;-x)+\Theta(x^{-1}\cdot q;-y)+2\Omega(x,y)]}L_\Omega(x+y)\otimes L_\Omega(x+y)$$

The phase factors are seen to be equal if we recall the expression for $\Omega(-y, -x)$ from (29) and the properties of Ω.

The coinvolution is projected to

$$\hat{\kappa}(L(x,\alpha)) \mapsto e^{i\Theta(q;-x)}\kappa^\Omega(L_\Omega(x))$$

where also $\hat{\kappa}$ is supposed not to act on the phase factor. From (13), using (34) and (35), we get

$$\kappa^\Omega(L_\Omega(x)) = e^{i[\Theta(q;x)-\Theta(q;-x)]}L_\Omega^\dagger(x) \tag{39}$$

Of all the axioms imposed on a coinvolution, κ^Ω fails to satisfy only one, the *antiautomorphism* axiom

$$\kappa^\Omega(L_\Omega(x)L_\Omega(y)) = \kappa^\Omega(L_\Omega(y))\kappa^\Omega(L_\Omega(x)) \tag{40}$$

By (39), and taking care of the phase factors in the operator products, we obtain from the left-hand side

$$\kappa^\Omega(L_\Omega(x)L_\Omega(y)) = e^{i[\Omega(x,y)+\Theta(q;x+y)-\Theta(q;-x-y)]}L_\Omega(-x-y) \tag{41}$$

while the right-hand side gives

$$\kappa^\Omega(L_\Omega(y))\kappa^\Omega(L_\Omega(x)) = e^{i[\Theta(q;y)-\Theta(q;-y)+\Theta(y\cdot q;x)-\Theta(y\cdot q;-x)+\Omega(-y,-x)]}L_\Omega(-x-y) \tag{42}$$

After using the explicit expressions for Ω and Θ, we get

$$\kappa^\Omega(L_\Omega(x)L_\Omega(y)) = e^{i(x_1y_2+y_1x_2)}\kappa^\Omega(L_\Omega(y))\kappa^\Omega(L_\Omega(x)) \tag{43}$$

instead of (40). We will return to this problem below. Concerning the remaining axioms that κ^Ω must satisfy: first, it should be involutive: $\kappa^\Omega(L_\Omega^\dagger(x)) = \kappa^\Omega(L_\Omega(x))^\dagger$. This follows from (39) and (34). The requirement $\kappa^\Omega(\kappa^\Omega(L_\Omega(x))) = L_\Omega(x)$ just implies that the phase factor in (39), which is antisymmetric in x, is canceled out when the second coinvolution is applied to $L_\Omega(-x)$. This is obvious. The *anticoautomorphism* axiom,

$$\Delta^\Omega\circ\kappa^\Omega = \sigma\circ(\kappa^\Omega\otimes\kappa^\Omega)\circ\Delta^\Omega \tag{44}$$

is clearly satisfied: when applied to $L_\Omega(x)$, the left-hand side of this equation raises the phase factor

$$e^{i[(\Theta(q;x)-\Theta(q;-x))+\Theta(q;x)]}$$

while the right-hand side raises the phase factor

$$e^{i[\Theta(q;-x)+2(\Theta(q;x)-\Theta(q;-x))]}$$

which is the same.

From (43) it is evident that the projection κ^{Ω} of the coinvolution κ is not a coinvolution on $\mathcal{M}^{\Omega}(\mathbb{R}^2)$. The role of a coinvolution in Kac duality is explicit in formula (20), the definition of the dual involution ^{o} on the predual of $\mathcal{M}(H_3)$. Since our main goal is to prove a duality for $\mathcal{M}^{\Omega}(\mathbb{R}^2)$, we are faced with a serious problem. The only weak aspect of the projection process of κ which could eventually be modified to solve this problem is the assumption that it does not act on the phase factor. But if it did act, the only plausible action would be by conjugation (35) (since it acts by *dagger* on L_{Ω}), and the resulting κ^{Ω} would be just $\kappa^{\Omega}(L_{\Omega}(x)) = L^{\dagger}_{\Omega}(x)$, the usual coinvolution of a symmetric group Kac algebra. In that case, it would not only fail to satisfy (40) but also the anticoautomorphism axiom, which involves the nontrivial Δ^{Ω}. More generally, if we define κ^{Ω} with any phase factor other than that of (39), say $e^{i\Psi(q;x)}$, the unique Ψ satisfying the last three axioms is just that combination of Θ's given in (39). This is most evident for the last axiom. We actually do not know of any good definition of κ^{Ω} making of it a coinvolution, that is, enforcing all the above axioms. The solution we have found for this problem is to maintain the definition of κ^{Ω} as it is given by the projection and modify the antiautomorphism axiom (40). A natural modification of it comes from the projection of the antiautomorphism axiom satisfied by $L(x, \alpha)$, which is given by

$$\hat{\kappa}(L(x, \alpha)L(y, \beta)) = \hat{\kappa}(L(y, \beta))\hat{\kappa}(L(x, \alpha)) \tag{45}$$

Using the projection formula (33) on it, we get, for example,

$$\hat{\kappa}(L(x, \alpha)L(y, \beta)) \mapsto e^{i[\Theta(q;-x)+\Theta(x^{-1}\cdot q;-y)]}\kappa^{\Omega}(L_{\Omega}(x)L_{\Omega}(y))$$

on its left-hand side. Doing the same with the other side, we find that (45) projects into

$$\kappa^{\Omega}(L_{\Omega}(x)L_{\Omega}(y)) \\ = e^{i[\Theta(q;-y)-\Theta(q;-x)+\Theta(y\cdot q;-x)-\Theta(x^{-1}\cdot q;-y)]}\kappa^{\Omega}(L_{\Omega}(y))\kappa^{\Omega}(L_{\Omega}(x)) \tag{46}$$

Given its nature, (46) should be called the *projective antiautomorphism* axiom. The importance of (46) comes from the fact that it is promptly satisfied by (39). In fact, substituting (41) and (42) in (46), we easily match the phase factors with the help of the expression (29) for Ω and of its properties. Notice that no new axiom arises if the other axioms defining a coinvolution are projected. This ends the list of axioms satisfied by what can now be called the *projective coinvolution* κ^{Ω}. It can be anticipated that the change from the axiom (40) to (46) will have consequences on the predual of $\mathcal{M}^{\Omega}(\mathbb{R}^2)$. The dual axiom, (44), for the dual coinvolution will be changed, too.

Finally, the trace $\hat{\varphi}$ is simply projected to its restriction to $\mathbb{R}^2$ according to

$$\hat{\varphi}(\hat{f}) = f(0, 1) \mapsto \varphi^{\Omega}(\hat{f}) = f(0) \tag{47}$$

where a general element $\hat{f}$ of $\mathcal{M}^{\Omega}(\mathbb{R}^2)$ is written

$$\hat{f} = \int_{\mathbb{R}^2} dx\, f(x) L_{\Omega}(x) \tag{48}$$

From its very definition, this trace is n.f.s. (see Section 4.1). It also satisfies the three specific axioms for a Haar weight, which are

$$(id \otimes \varphi^{\Omega})\Delta^{\Omega}(\hat{f}) = \varphi^{\Omega}(\hat{f})\mathbf{1} \qquad \forall \hat{f} \in \mathcal{M}^{\Omega}(\mathbb{R}^2)^+ \tag{49a}$$

$$(id \otimes \varphi^{\Omega})[(\mathbf{1} \otimes \hat{g}^{\dagger})\Delta^{\Omega}(\hat{f})] = \kappa^{\Omega} \circ (id \otimes \varphi^{\Omega})[\Delta^{\Omega}(\hat{g}^{\dagger})(\mathbf{1} \otimes \hat{f})] \tag{49b}$$

$$\kappa^{\Omega} \circ \sigma_t^{\varphi^{\Omega}} = \sigma_{-t}^{\varphi^{\Omega}} \circ \kappa^{\Omega} \qquad \forall t \in \mathbb{R} \tag{49c}$$

The third one is trivial here, since the modular group $\sigma^{\varphi^{\Omega}}$ is reduced to the identity when the Haar weight φ^{Ω} is a trace. Concerning the other axioms, we start by observing that, although φ^{Ω} is not defined on the generators $L_{\Omega}(x)$, from (47) and (48) it may be guessed that it would act on $L_{\Omega}(x)$ as $\varphi^{\Omega}(L_{\Omega}(x)) = \delta(x)$. This corresponds to an extension of the domain of φ^{Ω} to the generators, which can be regarded as being given by $L_{\Omega}(x) = \int_{\mathbb{R}^2} dy\, \delta_x(y) L_{\Omega}(y)$. We notice also that

$$\Delta^{\Omega}\hat{f} = \int_{\mathbb{R}^2} dx\, e^{i\Theta(q;-x)} f(x) L_{\Omega}(x) \otimes L_{\Omega}(x) \tag{50}$$

Axiom (49a) then follows from the considerations above, which imply $(id \otimes \varphi^{\Omega})\Delta^{\Omega}\hat{f} = f(0)L_{\Omega}(0) = \varphi^{\Omega}(\hat{f})\mathbf{1}$. In the same way the second follows from (39) and

$$\hat{f}^{\dagger} = \int_{\mathbb{R}^2} dx\, \overline{f(x)} L_{\Omega}^{\dagger}(x) \tag{51}$$

The resulting algebra, the projection of the symmetric Kac algebra of the Heisenberg group, will be denoted $\mathrm{K}^{\Omega}(\mathbb{R}^2)$ and called the *projective symmetric Kac algebra* of $\mathbb{R}^2$. It is built on the von Neumann algebra $\mathcal{M}^{\Omega}(\mathbb{R}^2)$ with the usual operator norm and conjugation. The remaining structures are grouped into

$$L_{\Omega}(x)L_{\Omega}(y) = e^{i\Omega(x,y)} L_{\Omega}(x + y) \tag{52a}$$

$$\mathbf{1} = L_{\Omega}(e) \tag{52b}$$

$$\Delta^{\Omega} L_{\Omega}(x) = e^{i\Theta(q;-x)} L_{\Omega}(x) \otimes L_{\Omega}(x) \tag{52c}$$

$$\kappa^{\Omega}(L_{\Omega}(x)) = e^{i[\Theta(q;x)-\Theta(q;-x)]} L_{\Omega}^{\dagger}(x) \tag{52d}$$

$$\varphi^{\Omega}(T) = \begin{cases} \|f\|_2^2 & \text{if } T = \hat{f}^{\dagger}\cdot\hat{f} \\ +\infty & \text{otherwise} \end{cases} \qquad T \in \mathcal{M}^{\Omega}(\mathrm{R}^2)^+ \tag{52e}$$

Comments

- The product of two elements $\hat{f}$, $\hat{g}$ is

$$\begin{aligned} \hat{f}\cdot\hat{g} &= \int_{\mathrm{R}^2\times\mathrm{R}^2} dx\, dy\, f(x)g(y)e^{i\Omega(x,y)}L_{\Omega}(x+y) \\ &= \int_{\mathrm{R}^2} dz \int_{\mathrm{R}^2} dx\, e^{i\Omega(x,z)}f(x)g(z-x)L_{\Omega}(z) \\ &= \int_{\mathrm{R}^2} dz\, (f \circledast g)(z)L_{\Omega}(z) \end{aligned} \tag{53}$$

where we have used the fact that Ω is antisymmetric to identify the twisted convolution

$$(f \circledast g)(z) = \int_{\mathrm{R}^2} dx\, e^{i\Omega(x,z)}f(x)g(z-x) \tag{54}$$

Since the operator product is mapped into the twisted convolution of L^1-functions, (48) can be regarded as the linear left-regular representation of $L^1_{\Omega}(\mathrm{R}^2)$ induced by L_{Ω}. Here $L^1_{\Omega}(\mathrm{R}^2)$ is the Banach algebra analogous to $L^1(\mathrm{R}^2)$, but with the twisted convolution. The involution remains in L^1 and its image by L_{Ω} gives the dagger of $\hat{f}$ [see (51)]: $\hat{f}^{\dagger} = L_{\Omega}(f^*)$.

- Despite the Abelian character of R^2, the projective product makes of $\mathrm{K}^{\Omega}(\mathrm{R}^2)$ a noncommutative algebra. The noncommutativity can be measured by the introduction of a Lie algebra structure on $\mathcal{M}^{\Omega}(\mathrm{R}^2)$ through the commutator

$$[L_{\Omega}(x), L_{\Omega}(y)] = 2i \sin[\Omega(x, y)]\, L_{\Omega}(x+y) \tag{55}$$

or by the introduction of the continuous R matrix

$$L_{\Omega}(x)L_{\Omega}(y) = \int_{\mathrm{R}^2\times\mathrm{R}^2} dz\, dw\, R(x, y; z, w)L_{\Omega}(z)L_{\Omega}(w)$$

The R matrix elements belong to $M^1_{\Omega}(\mathrm{R}^2) \supset L^1_{\Omega}(\mathrm{R}^2)$ and are given by

$$R(x, y; z, w) = e^{i[\Omega(x,y)-\Omega(z,w)]}\delta(x+y-w-z) \tag{56}$$

Associativity of (52a) implies that R should satisfy the Yang–Baxter equation, and (56) provides a new nontrivial solution for it. Let us observe that the associativity of (52a), the Jacobi identity for the commutator (55), and the Yang–Baxter equation depend on the closedness of Ω.

- The ideal of elements such that $\varphi^{\Omega}(\hat{f}^{\dagger}\hat{f}) < \infty$ is just $\mathcal{N}_{\varphi^{\Omega}} = L^1_{\Omega} \cap L^2(\mathbb{R}^2)$. So, the GNS representation of this projective Kac algebra is given on $L^2(\mathbb{R}^2)$ by $\pi_{\varphi^{\Omega}}(\hat{f})g = f \circledast g$. The GNS image in L^2 of $\hat{f}$ will be denoted $\hat{f}_{\varphi^{\Omega}}$.
- We can recover the L^1_{Ω}-function f in the linear combination (48) through the Haar trace φ^{Ω} by the formula

$$f(x) = \varphi^{\Omega}[L^{\dagger}_{\Omega}(x)\hat{f}]$$

- Written in terms of the M^1_{Ω}-distributions ($L^1_{\Omega} \subset M^1_{\Omega}$), the projective Kac algebra operations, other than the twisted convolution and the trace, read ($f \in L^1_{\Omega}$)

$$\Delta^{\Omega}(f)(x, y) = e^{i\Theta(q;-x)}f(x)\delta(x - y) \tag{57a}$$

$$\kappa^{\Omega}(f)(x) = e^{-i[\Theta(q;x)-\Theta(q;-x)]}f(-x) \tag{57b}$$

5.1.2. *The Dual Algebra of* $\mathbb{K}^{\Omega}(\mathbb{R}^2)$

Analogous to what was done for the Heisenberg group, we now start fixing the predual of $\mathcal{M}^{\Omega}(\mathbb{R}^2)$ and, by its Fourier representation, the dual of $\mathbb{K}^{\Omega}(\mathbb{R}^2)$. Even though the latter is not a Kac algebra, the same techniques for establishing a duality for it will be used. In this section L^p will denote $L^p(\mathbb{R}^2)$ for $p = 1, 2, \infty$.

The elements of $\mathcal{M}^{\Omega}(\mathbb{R}^2)_*$ will be written as linear forms ω^{Ω}_{fg} on $\mathcal{B}(L^2)$ whose duality pairing with $L^{\dagger}_{\Omega}(x)$ gives the functions

$$\langle L^{\dagger}_{\Omega}(x), \omega^{\Omega}_{fg}\rangle = (L_{\Omega}(-x)f \mid g)_{L^2} = (f \circledast \check{g})(x), \qquad f, g \in L^2$$

The functions $\omega^{\Omega}_{fg}(x) \equiv (f \circledast \check{g})(x)$ are thereby defined as the representative functions in the predual. From this definition it also turns out that these functions are in L^{∞}: $|\omega^{\Omega}_{fg}(x)| \leq \|f\|_2\|g\|_2 < \infty$. When no confusion can arise, they will be denoted simply by f, g, h, etc.

The product on $\mathcal{M}^{\Omega}(\mathbb{R}^2)_*$ will follow from duality by

$$\begin{aligned}(\omega^{\Omega}_{fg} \star \omega^{\Omega}_{hl})(x) &= \langle \Delta^{\Omega}L_{\Omega}(-x), \omega^{\Omega}_{fg} \otimes \omega^{\Omega}_{hl}\rangle \\ &= e^{i\Theta(q;x)}(L_{\Omega}(-x) \otimes L_{\Omega}(-x)(f \otimes h) \mid g \otimes l)_{L^2\otimes L^2} \\ &= e^{i\Theta(q;x)}\omega^{\Omega}_{fg}(x)\omega^{\Omega}_{hl}(x)\end{aligned} \tag{58}$$

and, since Δ^{Ω} is symmetric, $\star$ is Abelian. Its associativity follows trivially. The unit is a consequence of (58) and is uniquely given by

$$\mathbf{1}(x) = e^{-i\Theta(q;x)}$$

To finish with the characterization of this predual, the dual involution ^{o} is found from

$$\begin{aligned}(\omega_{fg}^{\Omega})^o(x) &= \langle L_{\Omega}(-x), \omega_{fg}^{\Omega\, o}\rangle \\ &= \overline{\langle \kappa^{\Omega}(L_{\Omega}^{\dagger}(-x)), \omega_{fg}^{\Omega}\rangle} \\ &= e^{-i[\Theta(q;x)-\Theta(q;-x)]}\overline{\omega_{fg}^{\Omega}(x)} \qquad (59)\end{aligned}$$

It is indeed an involution, since it is antilinear, satisfies $^o \circ {}^o = id$, $\mathbf{1}^o(x) = \mathbf{1}(x)\ \forall x$, and is an antiautomorphism because, after recalling (35), the following two equations turn out to be equal:

$$\begin{aligned}(f \star g)^o(x) &= e^{-i[\Theta(q;x)-\Theta(q;-x)]}\overline{(e^{i\Theta(q;x)})^* f(x)g(x)} \\ (g^o \star f^o)(x) &= e^{i\Theta(q;x)-2[\Theta(q;x)-\Theta(q;x)]}\overline{g(x)f(x)}\end{aligned}$$

In analogy with the Kac algebra case, we call the predual $\mathcal{M}^{\Omega}(\mathrm{R}^2)_*$ the *projective Fourier algebra* of R^2 and denote it by $A^{\Theta}(\mathrm{R}^2)$. Since their elements are L^{∞}, we will denote the respective von Neumann algebra by $L^{\infty}_{\Theta}(\mathrm{R}^2)$. As a final remark, notice that the projection of the product $(f \cdot g)(x, \alpha) = f(x, \alpha)g(x, \alpha)$, giving the correct projected product (58), comes from the projection of $f \in L^{\infty}(H_3)$ to $L^{\infty}_{\Omega}(\mathrm{R}^2)$ in the following way:

$$f(x, \alpha) \mapsto e^{-i\Theta(q;-x)}f(x) \qquad (60)$$

Actually, it gives the $\star$ product if we allow the phase Θ to *feel* the action of R^2 on R as follows: $f(x, \alpha)g(y, \beta) = (f \otimes g)(x, \alpha \otimes y, \beta)$ is projected, after (60) and considering that y acts on the phase at x, to $e^{-i\Theta(y \cdot q;-x)}e^{-i\Theta(q;-y)}f(x)g(y)$, while $(f \cdot g)(x, \alpha)$ goes to $e^{-i\Theta(q;-x)}(f \star g)(x)$. Making $(y, \beta) = (x, \alpha)$ and recalling that $-\Theta(x \cdot q; -x) = \Theta(q; x)$, we find that the exponentials without action cancel out and the correct $\star$ product is obtained. In the same way, the action of $L^{\infty}(H_3)$ on the Hilbert space $H(H_3)$ by pointwise product is projected to the action of $L^{\infty}_{\Theta}(\mathrm{R}^2)$ on $L^2(\mathrm{R}^2)$ by the $\star$ product. Projection (60) is also useful to project L^1-functions. From (60) and (33) we get the right projection of the $\mathrm{K}^s(H_3)$-elements (10) into the operators (48).

The Fourier representation λ^{Ω} of $A^{\Theta}(\mathrm{R}^2)$ is defined, in analogy with (21), by

$$[\lambda^{\Omega}(\omega^{\Omega})f](x) = [(\omega^{\Omega} \circ \kappa^{\Omega} \otimes id)\Delta^{\Omega}\hat{f}]_{\varphi^{\Omega}}(x), \qquad f = \hat{f}_{\varphi^{\Omega}} \in L^2 \cap L^1_{\Omega} \qquad (61)$$

From (50) and (52d) we readily get

$$(\omega^\Omega \circ \kappa^\Omega \otimes id)\Delta^\Omega \hat{f} = (\omega^\Omega \otimes id)\int_{\mathbf{R}^2} dx\, e^{i\Theta(q;x)} f(x) L_\Omega(-x) \otimes L_\Omega(x)$$

$$= \int_{\mathbf{R}^2} dx\, e^{i\Theta(q;x)} f(x)\omega^\Omega(x) L_\Omega(x)$$

which gives

$$\lambda^\Omega(\omega^\Omega) f(x) = (\omega^\Omega \star f)(x)$$

Since the action of $A^\Theta(\mathbf{R}^2)$ [or $L^\infty_\Theta(\mathbf{R}^2)$] is given by the $\star$ product, it follows that $\lambda^\Omega = id$. This means that, as in the Kac case, the dual of $\mathrm{K}^\Omega(\mathbf{R}^2)$ is built on the von Neumann algebra $L^\infty_\Theta(\mathbf{R}^2)$, the L^∞-Banach algebra with the product $\star$, and the involution ^{o} of $A^\Theta(\mathbf{R}^2)$. Its norm is the same as that of L^∞ and satisfies $\|f^o\| = \|f\|$, $\|f^o \star f\| = \|f\|^2$.

The generator of λ^Ω, that is, the operator W^Θ in $L^\infty_\Theta \otimes \mathcal{B}(L^2)$ satisfying $\lambda^\Omega(\omega^\Omega) = (id \otimes \omega^\Omega)(W^\Theta)$, is given by

$$[W^\Theta F](x, y) = e^{i\Theta(q;x)} e^{-i\Omega(x,y)} F(x, y + x) \tag{62}$$

We see that this generator acts on $(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2$ as if it were $W^\Theta \sim 1 \otimes L^\dagger_\Omega(x)$, with 1 the constant function $1 \in L^\infty_\Theta$.

By duality, we obtain also a coproduct on the dual $L^\infty_\Theta(\mathbf{R}^2)$:

$$\begin{aligned}\Delta^\Theta(\omega^\Omega_{fg})(x, y) &= \langle [L_\Omega(x) L_\Omega(y)]^\dagger, \omega^\Omega_{fg}\rangle \\ &= e^{-i\Omega(x,y)}(L^\dagger_\Omega(x + y) f \mid g) \\ &= e^{-i\Omega(x,y)}\omega^\Omega_{fg}(x + y)\end{aligned} \tag{63}$$

This coproduct is automatically coassociative, as it is defined by the associative dual product. It is also unital: $\Delta^\Theta \mathbf{1}(x, y) = e^{-i\Omega(x,y)} e^{-i\Theta(q;x+y)}$, while $(\mathbf{1} \otimes \mathbf{1})(x, y) = e^{-i[\Theta(y\cdot q;x)+\Theta(q;y)]}$. In this last expression it must be recalled that the phase factor $e^{i\Theta}$ is a special kind of complex function which is *sensitive* to the action of $\mathbf{R}^2$ on $\mathbf{R}$, even when the simple product in $\mathbf{C}$ is performed. The homomorphism axiom was already proved for $\mathrm{K}^\Omega(\mathbf{R}^2)$, but it is interesting to verify it here again, since it confirms the strange behavior of $e^{i\Theta}$ under the complex product. It follows from

$$\Delta^\Theta(f \star g)(x, y) = e^{-i\Omega(x,y)} e^{i\Theta(q;x+y)} f(x + y) g(x + y)$$

$$\begin{aligned}(\Delta^\Theta f \star \Delta^\Theta g)(x, y) &= (\Delta^\Theta f^{(1)} \star \Delta^\Theta g^{(1)})(x)(\Delta^\Theta f^{(2)} \star \Delta^\Theta g^{(2)})(y) \\ &= e^{i\Theta(y\cdot q;x)} e^{i\Theta(q;y)} \Delta^\Theta f(x, y) \Delta^\Theta g(x, y) \\ &= e^{i\Theta(y\cdot q;x)} e^{i\Theta(q;y)} e^{-2i\Omega(x,y)} f(x + y) g(x + y)\end{aligned}$$

where we have written $\Delta^\Theta f = \Delta^\Theta f^{(1)} \otimes \Delta^\Theta f^{(2)}$. The homomorphism is established after recalling the expression for $-\Omega(x, y)$. The above two axioms would not hold if we did not allow Θ to *feel* the action of $\mathbb{R}^2$ on $\mathbb{R}$.

The candidate coinvolution in L^∞_Θ comes from the duality relation

$$\kappa^\Theta(\omega^\Omega_{fg})(x) = \langle \kappa^\Omega(L_\Omega(-x)), \omega^\Omega_{fg}\rangle = e^{-i[\Theta(q;x)-\Theta(q;-x)]}\omega^\Omega_{fg}(-x) \qquad (64)$$

It satisfies the first three coinvolution axioms without any problem, including the antiisomorphism condition, which its dual κ^Ω fails to satisfy. The problem lies precisely in the anticoautomorphism axiom. This is just the dual of the problem found in $\mathbb{K}^\Omega(\mathbb{R}^2)$, and its solution will be given by dualizing the solution of that problem, namely dualizing the axiom (46). Recall that (40) can be written in the form $\kappa^\Omega \circ m = m \circ (\kappa^\Omega \otimes \kappa^\Omega) \circ \sigma$, where m denotes the product on $\mathbb{K}^\Omega(\mathbb{R}^2)$. We adapt this form to axiom (46) and dualize it, that is, just transpose the order and change the operations to their duals, taking into account the effect (35) of duality on Θ and transposing $x \leftrightarrow y$. It should be noticed that, from the properties of Θ, it follows also that

$$(e^{i\Theta(x^{-1}\cdot q;y)})^* = e^{i\Theta(x\cdot q;-y)}$$

We obtain

$$[\Delta^\Theta \circ \kappa^\Theta f](x, y) = e^{-i[\Theta(q;y)-\Theta(q;x)+\Theta(y\cdot q;x)-\Theta(x^{-1}\cdot q;y)]}[\sigma \circ (\kappa^\Theta \otimes \kappa^\Theta) \circ \Delta^\Theta f](x, y) \quad (65)$$

as a new axiom replacing (44). It will be called the *projective anticoautomorphism* axiom. Provided care is taken with the product of complex functions $e^{i\Theta}$ when computing its right-hand side, (64) is promptly seen to satisfy this new axiom. After these changes κ^Θ should be called the *coprojective coinvolution.*

Finally, the trace (23e) is projected to the n.f.s. trace

$$\varphi(f) = \int_{H_3} dx\, d\alpha\, f(x, \alpha) \mapsto \varphi^\Theta(f) = \int_{\mathbb{R}^2} dx\, e^{-i\Theta(q;-x)} f(x) \qquad (66)$$

where $f \in L^\infty_\Theta(\mathbb{R}^2)^+$. If this projection is interpreted as coming from the projection (60) of $f \in L^\infty(H_3)$ to $L^\infty_\Theta(\mathbb{R}^2)$, it is evident that the projection of φ to φ^Θ is only possible due to the compactness of the central group $\mathbb{T}$. This fact has also been observed in prequantization (Tuynman and Wiegerinck, 1987).

The trace (66) is of Haar type, since it satisfies the axiom (49b), for example, as follows: using the formulas for φ^Θ, ^{o}, and $\star$, we obtain from the left-hand side

$$
\begin{aligned}
&(id \otimes \varphi^\Theta)[(\mathbf{1} \otimes g^o) \star \Delta^\Theta f](x) \\
&= \int_{\mathbb{R}^2} dy\, e^{-i\Theta(q;-y)} e^{i\Theta(q;y)} g^o(y) \Delta^\Theta f(x, y) \\
&= \int_{\mathbb{R}^2} dy\, \Delta^\Theta f(x, y) \overline{g(y)}
\end{aligned}
$$

while from the right-hand side we have

$$
\begin{aligned}
&\kappa^\Theta\{(id \otimes \varphi^\Theta)[\Delta^\Theta g^o \star (\mathbf{1} \otimes f)]\}(x) \\
&= e^{-i[\Theta(q;x)-\Theta(q;-x)]}(\Delta^\Theta g^o)^{(1)}(-x)\varphi^\Theta[(\Delta^\Theta g^o)^{(2)} \star f] \\
&= \int_{\mathbb{R}^2} dz\, \overline{\Delta^\Theta g(-x, z)} f(z)
\end{aligned}
$$

They become equal when we substitute the expression for the coproduct Δ^Θ and make the change of variable $y = z - x$ in the last integral. Axiom (49a) follows similarly and (49c) results automatically because σ^{φ^Ω} reduces to the identity.

The algebra so far obtained will be called the *projective Abelian Kac algebra* of $\mathbb{R}^2$ and will be denoted $K^\Theta(\mathbb{R}^2)$. It is built on $L^\infty_\Theta(\mathbb{R}^2)$, and its structure, besides the usual L^∞-norm, is summed up in the following properties:

$$f^o(x) = e^{-i[\Theta(q;x)-\Theta(q;-x)]}\overline{f(x)} \tag{67a}$$

$$(f \star g)(x) = e^{i\Theta(q;x)} f(x) g(x) \tag{67b}$$

$$\mathbf{1}(x) = e^{-i\Theta(q;x)} \tag{67c}$$

$$\Delta^\Theta f(x, y) = e^{-i\Omega(x,y)} f(x + y) \tag{67d}$$

$$\kappa^\Theta f(x) = e^{-i[\Theta(q;x)-\Theta(q;-x)]} f(-x) \tag{67e}$$

$$\varphi^\Omega(f) = \int_{\mathbb{R}^2} dx\, e^{-i\Theta(q;-x)} f(x) \tag{67f}$$

Comments

- The name *projective* does not put $K^\Omega(\mathbb{R}^2)$ and $K^\Theta(\mathbb{R}^2)$ into the same category, since the *projective* coinvolution κ^Ω and the *coprojective* coinvolution κ^Θ are defined by different sets of axioms.
- It can be confirmed that $L^\infty_\Theta(\mathbb{R}^2)$ acts on $L^2(\mathbb{R}^2)$ from the GNS construction induced by the trace φ^Θ. This trace defines a scalar product on the ideal of elements f such that $\varphi^\Theta(f^o \star f) < \infty$ by $(f|g) \equiv \varphi^\Theta(g^o \star f)$. Direct calculation from (67a) and (67b) gives the usual L^2-scalar product $(f|g) =$

$\int_{\mathbf{R}^2} dx,\, f(x)\overline{g(x)}$. The inclusion of $f \in L^\infty_\Theta$ into L^2 by this construction is denoted f_{φ^Θ}.

The scalar product defined by φ^Θ also shows that the involution ^{o} goes to the simple complex conjugation in L^2. Thus, the antiunitary operators implementing on L^2 the involutions ^{o} and $*$ are simply

$$J^\Theta f(x) = \overline{f(x)}$$
$$J^\Omega f(x) = \overline{f(-x)}$$

They provide the adjoint of the unitary Fourier representation generators, for example,

$$W^{\Theta *} = (J^\Omega \otimes J^\Theta) W^\Theta (J^\Omega \otimes J^\Theta) \tag{68}$$

whose action on $L^2 \times L^2$ is

$$[W^{\Theta *}F](x, y) = e^{i\Theta(q;-x)} e^{i\Omega(x,y)} F(x, y - x)$$

Operator J^Ω is the correct projection of $\hat{J}$, but J^Θ is just the implementation of ^{o} as complex conjugation and does not equal the projection of $[Jf](x, \alpha) = f(x, \alpha)$. The latter is projected to

$$J'^\Theta f(x) = e^{-i[\Theta(q;x) - \Theta(q;-x)]} \overline{f(x)}$$

In terms of the actual projected operators we have the canonical implementations of κ^Ω and κ^Θ in L^2:

$$\kappa^\Omega(L_\Omega(x)) = J'^\Theta L^\dagger_\Omega(x) J'^\Theta$$
$$\kappa^\Theta(f) = J^\Omega f^o J^\Omega$$

That κ^Ω is implemented by J'^Θ and not by J^Θ can be explained if we recall that (1) in contrast to κ^Θ, κ^Ω is not a linear antiautomorphism, but a projective one, and (2) the involutions embodied in the antiunitary operators J are closely related to the algebra product and not to the coproduct.

As regards the canonical implementation of Δ^Θ and Δ^Ω, it is easily verified that W^Θ and W^Ω, respectively, do the job. After recalling the $\star$-action of L^∞_Θ on L^2, we find that

$$[W^\Theta(1 \otimes f) W^{\Theta *} F](x, y)$$
$$= e^{i[\Theta(q;x) + \Theta(q;x+y) + \Theta(x \cdot q; -x)]} f(x + y) F(x, y)$$

The phase $\Theta(x \cdot q;\, -x)$ comes from the action of $W^{\Theta *}$ over the previous action of W^Θ by $L_\Omega(-x)$ and cancels out with the first Θ. On the other hand, we have

$$[\Delta^{\Theta}(f)F](x, y) = e^{-i[\Omega(x,y)+\Theta(y\cdot q;x)+\Theta(q;y)]}f(x + y)F(x, y)$$

Recalling the expression for Ω as the cohomological derivative of Θ, we arrive at the equality of the left-hand sides, namely

$$\Delta^{\Theta}(f) = W^{\Theta}(1 \otimes f)W^{\Theta *} \tag{69}$$

This implementation is not unique, as (69) will be also satisfied for any operator V^{Θ} introduced through $W^{\Theta}F(x, y) = e^{i\Theta(q;x)}V^{\Theta}F(x, y)$ or, equivalently, by $V^{\Theta}F(x, y) = e^{-i\Omega(x,y)}F(x, y + x)$. Nevertheless, W^{Θ} is the unique operator generating the Fourier representation λ^{Ω}. In Kac algebra terminology, it is the *fundamental operator* of the projective Kac algebra $K^{\Theta}(R^2)$. This means that it satisfies the pentagonal relation, a point easily confirmed by direct calculation.

To show the implementation of Δ^{Ω}, we first find $W^{\Omega} = \widehat{W^{\Theta}} = \sigma \circ W^{\Theta *} \circ \sigma$ and $W^{\Omega *} = \sigma \circ W^{\Theta} \circ \sigma$,

$$W^{\Omega}F(x, y) = e^{i\Omega(y,x)}e^{i\Theta(q;-y)}F(x - y, y) \tag{70a}$$

$$W^{\Omega *}F(x, y) = e^{-i\Omega(-y,x)}e^{i\Theta(q;y)}F(x + y, y) \tag{70b}$$

Keeping in mind that these generators behave like $W^{\Omega} \sim L_{\Omega}(y) \otimes 1^{o}$ and $W^{\Omega *} \sim L_{\Omega}^{\dagger}(y) \otimes 1$, we get, after some cancellations,

$$[W^{\Omega}(I \otimes L_{\Omega}(z))W^{\Omega *}F](x, y)$$
$$= e^{i[\Omega(z,x)+\Theta(q;-y)+\Theta(y^{-1}\cdot q;y-z)]}F(x - z, y - z)$$

and

$$\Delta^{\Omega}L_{\Omega}(z)F(x, y) = e^{i[\Theta(q;-z)+\Omega(z,x)+\Omega(z,y)]}F(x - z, y - z)$$

which turn out to be equal if we recall that $\Omega(z, y) = \Omega(y - z, -y)$. Unlike what happens in the dual case, the operator defined from the expression for W^{Ω} by $V^{\Omega}F(x, y) = e^{-i\Omega(x,y)}F(x - y, y)$ does not implement Δ^{Ω}.

In order to establish a projective duality, we proceed now to determine the predual of L^{∞}_{Θ} and its Fourier representation. First, we must point out a particularity of projective algebras: the duality pairing $\langle \cdot, \cdot \rangle$ used so far to connect $\mathcal{M}^{\Omega}(R^2)$ and its predual $A^{\Theta}(R^2)$ involves implicitly a complex conjugation of the phase factors, since the representative functions in the projective Fourier algebra are defined in (18) by pairing with $L_{\Omega}^{\dagger}$ and not with L_{Ω}. The result is that the phase factors in the structure of $A^{\Theta}(R^2)$ get complex-conjugated. Furthermore, the dualization process from (66), which brings the projective antiautomorphism axiom from $K^{\Omega}(R^2)$ to the projective anticoautomorphism axiom of $K^{\Theta}(R^2)$, involves not only a transposition, but also a complex conjugation of the phase factors. In contrast to these duality pairings, the impossibility of working with the L^{∞}_{Θ}-generators forces us to put generic L^∞_Θ-functions g in the duality pairing between this algebra and its predual, which is usually given by $\langle g, f\rangle = \int_{\mathbb{R}^2} dx\, g(x)f(x)$. That is, the complex conjugation of the phase factors is *implicit*. With these remarks in mind, and recalling that the Θ-phase factors are complex-conjugated according to (35), while those involving Ω are complex-conjugated as usual, we begin by introducing the representative functions on $L^\infty_{\Theta *}$ through

$$\langle g, \omega^\Theta_{hf}\rangle = (g \star h | f) = \int dx\, g(x) e^{i\Theta(q;-x)} h(x)\overline{f(x)}, \qquad g \in L^\infty_\Theta, \quad h, f \in L^2$$

$$\therefore\ \omega^\Theta_{hf}(x) = e^{i\Theta(q;-x)} h(x)\overline{f(x)} = (h \star f^o)(x) \tag{71}$$

When no confusion can arise, these functions will be also denoted by f, g, h, etc. The product in the predual comes from the coproduct Δ^Θ by

$$\langle g, f \circledast h\rangle = \langle \Delta^\Theta g, f \otimes h\rangle = \int_{\mathbb{R}^2\times\mathbb{R}^2} dx\, dy\, e^{i\Omega(x,y)} g(x+y) f(x) h(y)$$

which gives the twisted convolution (54). The involution * comes from

$$\langle g, f^*\rangle = \overline{\langle \kappa^\Theta(g^o), f\rangle} = \int_{\mathbb{R}^2} dx\, g(x)\overline{f(-x)}$$

and coincides with the L^1-involution. At this point we could already guess that the predual we are looking for is just $L^1_\Omega(\mathbb{R}^2)$. This is confirmed when we recall the Hölder inequality (Choquet-Bruhat *et al.*, 1982) for L^p-spaces, $p = 1, 2$, which says that, if $h, f \in L^2$, then the modulus of their product is an integrable function in L^1. This implies that the function ω^Θ_{hf} given in (71) are L^1. Furthermore, their product and involution also characterize them as $L^1_\Omega(\mathbb{R}^2)$-functions.

If duality is to hold, the dual of the Fourier representation λ^Ω should be generated by W^Ω, the dual of W^Θ. This dual generator has already been given in (70a). The representation of $L^1_\Omega(\mathbb{R}^2)$ it generates is denoted by λ^Θ and follows from the identity

$$(g | \lambda^\Theta(\omega^\Theta_{hf}) \star l) = (g \otimes f | W^\Omega(l \otimes h))$$

Recalling that $\omega^\Theta_{hf} = h \circledast \check{f} \in L^1_\Omega$, we find that λ^Θ is given by

$$\lambda^\Theta(\omega^\Theta) = \int_{\mathbb{R}^2} dx\, \omega^\Theta(x) L_\Omega(x) \tag{72}$$

Its range as an operator on a Hilbert space is restricted to $L^\infty_\Theta \cap L^2(\mathbb{R}^2)$. Let us observe that λ^Θ *cannot* be written

$$\lambda^\Theta(\omega^\Theta)f_{\varphi^\Theta} = [(\omega^\Theta \circ \kappa^\Theta \otimes id)(\Delta^\Theta)(f)]_{\varphi^\Theta}$$

perhaps because κ^Θ is not an anticoautomorphism.

Formula (72) coincides with the expression (48) for a generic element of $\mathcal{M}^\Omega(\mathbb{R}^2)$, which enables us to conclude, from (53) and (51), that it is actually a linear and involutive representation of $L^1_\Omega(\mathbb{R}^2)$ in that von Neumann algebra. At this point it is no longer necessary to show that the product $\star$ and the coprojective coinvolution κ^Θ go, by duality, respectively into the coproduct Δ^Ω and the projective coinvolution κ^Ω. Furthermore, in addition to being the unique operator implementing the coproduct Δ^Ω and generating λ^Θ, the fundamental operator W^Ω satisfies the pentagonal relation. All these facts confirm the existence of a duality between the projective Kac algebras $\mathbb{K}^\Omega(\mathbb{R}^2)$ and $\mathbb{K}^\Theta(\mathbb{R}^2)$. By the association of these symmetric and Abelian projective Kac algebras to the Abelian group $\mathbb{R}^2$, the projective Kac duality provides a *projective Fourier duality* for this group.

As a by-product of the pentagonal relation, which can be considered as *the symbol of duality* (Baaj and Skandalis, 1993) and is satisfied by both W^Θ and W^Ω, we find that the operators V^Θ and V^Ω, coming from

$$W^\Theta F(x, y) = e^{i\Theta(q;x)}V^\Theta F(x, y)$$

$$W^\Omega F(x, y) = e^{i\Theta(q;-y)}V^\Omega F(x, y)$$

satisfy the following *projective* versions of that relation:

$$[V^\Theta_{23}V^\Theta_{13}V^\Theta_{12}F](x, y, z) = e^{i\Omega(x,y)}[V^\Theta_{12}V^\Theta_{23}F](x, y, z)$$

$$[V^\Omega_{23}V^\Omega_{13}V^\Omega_{12}F](x, y, z) = e^{-i\Omega(y,z)}[V^\Omega_{12}V^\Omega_{23}F](x, y, z)$$

As regards projective Kac duality, it must be remembered that these algebras are not objects in the same category if the Kac algebra category definition given in Enock and Schwartz (1992) is to be maintained. The coinvolutions in the projective Kac algebras satisfy different sets of axioms. They would become objects of the same category if we could define a wider category whose objects would be algebras similar to Kac algebras, but where the coinvolutions would be more general linear maps κ' satisfying only the axioms

$$\kappa' \circ * = * \circ \kappa'$$

$$\kappa' \circ \kappa' = id$$

Unfortunately, such a category is not well defined, since the anticoautomorphism property of κ^Ω and the antiautomorphism property of κ^Θ seem to play an important role in projective Kac duality. For example, the property of κ^Ω alluded to seems to be responsible for the expression (61), while the same is not true between κ^Θ and Δ^Θ.

5.2. Irreducible Decomposition According to the Projective Dual

This subsection is devoted to the decomposition of the projective Kac duality obtained in the last subsection according to the projective unitary dual of $\mathbb{R}^2$. We begin with the decomposition of the projective Kac algebra $\mathbb{K}^{\Omega}(\mathbb{R}^2)$ according to the decomposition of the left-regular representations L_{Ω} in terms of projective irreducible representations. The latter are obtained by restricting to $\mathbb{R}^2$ the irreducible linear representations of H_3 shown in (8). The result is

$$[S_\nu(x)\xi](q) = e^{-i\nu\Theta(q;-x)}\xi(q - x_1), \qquad \nu \in \mathbb{Z} - \{0\} \tag{73}$$

By direct calculation and using the relation between the cochains Θ and Ω, one easily verifies that these operators satisfy the projective relation (5), while the members of the other series of irreducible representations of H_3, (8b), do not. By the Stone–von Neumann theorem and Bargmann's method, we conclude that (73) are the unique irreducible projective representations of the plane group. Since they are also inequivalent, the Ω-projective dual of $\mathbb{R}^2$, here denoted $\widehat{\mathbb{R}^2_\Omega}$, is just $\mathbb{Z} - \{0\}$. The von Neumann algebra generated by this kind of bounded operator on $L^2(\mathbb{R})$ will be denoted $\mathcal{M}^{\Omega}_{\nu}(\mathbb{R}^2)$. Since the operators (73) come from the representations of the Heisenberg group, which is a group of type I, the algebra $\mathcal{M}^{\Omega}_{\nu}(\mathbb{R}^2)$ is also of type I. In the following we will proceed along the lines of the symmetric Kac algebra decomposition exposed in Aldrovandi and Saeger (1996). The task here will be simpler than in that work, since the Haar weight involved is a trace. The decomposition of a von Neumann algebra generated by regular representations of a unimodular type I group was already established in Dixmier (1977). The only new aspect here is that the representations involved are *projective*. To proceed further, we will suppose the existence of a positive measure $\mu(\nu)$ on $\widehat{\mathbb{R}^2_\Omega}$ such that the following equality holds true:

$$L_\Omega = \sum_{\nu\in \mathbb{Z}-\{0\}}^{\oplus} \mu(\nu) S_\nu \tag{74}$$

The decomposition (74) is based on the facts that (i) both L_Ω and S_ν are projective operators, (ii) these are irreducible, and (iii) the representations (73) can also be defined by

$$[S_\nu(x)f_\nu](y) = e^{i\nu\Omega(x,y)}f_\nu(y - x) \tag{75}$$

where the functions $f_\nu \in H_\nu(\mathbb{R}^2) \sim L^2(\mathbb{R})$ enter in the projective decomposition of $f \in L^2(\mathbb{R}^2)$ according to $f = \Sigma_{\nu\in\mathbb{Z}-\{0\}}\ \mu(\nu)f_\nu$ and are given by

$$f_\nu(x) = e^{(i\nu/2)x_1x_2}\xi(x_1)$$

Taking this Hilbert-space projective decomposition into (75), we promptly obtain (73), with $q = y_1$. Thus, S_ν is actually an irreducible projective component of L_Ω.

Since (74) does make sense, we can go on and decompose the respective representation of $L^1_\Omega(\mathbf{R}^2)$ through

$$L_\Omega(f) = \sum_{\nu \in \mathbf{Z}-\{0\}} \mu(\nu) S_\nu(f)$$

where

$$S_\nu(f) \equiv \hat{f}_\nu = \int_{\mathbf{R}^2} dx\, f(x) S_\nu(x) \tag{76}$$

The last formula can be regarded as the *projective Fourier transform* on $\mathbf{R}^2$, that is, a map associating an operator-valued function of $\widehat{\mathbf{R}^2_\Omega}$ to each L^1_Ω-function on $\mathbf{R}^2$.

Operators (76) act on $L^2(\mathbf{R})$ according to (73), through

$$[\hat{f}_\nu \xi](q) = \int_{\mathbf{R}} du\, K_f^\nu(q, u) \xi(u)$$

where the kernel K_f^ν is given by

$$K_f^\nu(q, u) = \int_{\mathbf{R}} dv\, e^{-i\nu\Theta(q;(u-q,-v))} f(q - u, v)$$

This enables us to introduce a trace on the operators (76) by

$$\mathrm{Tr}_\nu(\hat{f}_\nu) \equiv \frac{1}{2\pi\nu} \int_{\mathbf{R}} dq\, K_f^\nu(q, q)$$

After recognizing the Dirac delta distribution on $\mathbf{R}$,

$$\delta(\nu) = \frac{1}{2\pi} \int_{\mathbf{R}} dq\, e^{iq\nu}$$

we find that the above trace turns out to be

$$\mathrm{Tr}_\nu(\hat{f}_\nu) = f(0) \tag{77}$$

for all $\nu \in \mathbf{Z} - \{0\}$. On the other hand, by Dixmier (1977), the decomposition of the trace φ^Ω on $\mathrm{K}^\Omega(\mathbf{R}^2)$ should be given by n.f.s. traces φ^Ω_ν according to

$$\varphi^\Omega(\hat{f}) = \sum_{\nu \in \mathbf{Z}-\{0\}} \mu(\nu) \varphi^\Omega_\nu(\hat{f}_\nu) \tag{78}$$

If we take $\varphi^\Omega_\nu = \mathrm{Tr}_\nu$ and recall that $\varphi^\Omega(\hat{f}) = f(0)$, we conclude from (77) that the measure μ must be such that $\Sigma_{\nu \in \mathbf{Z}-\{0\}}\, \mu(\nu) = 1$. As a first application of the trace decomposition, we establish a projective version of the Plancherel formula. It comes as a consequence of $\varphi^{\Omega}(\hat{f}^{\dagger}\hat{f}) = (f^* \circledast f)(0)$ and of the decomposition (78),

$$\int_{\mathbb{R}^2} dx\, |f(x)|^2 = \sum_{\nu \in \mathbb{Z}-\{0\}} \mu(\nu)\, \mathrm{Tr}_\nu[\hat{f}_\nu^{\dagger}\hat{f}_\nu] \tag{79}$$

which allows μ to be called the *projective Plancherel measure* associated to the Haar measure on $\mathbb{R}^2$.

We recover the L^1_Ω-function at (76), or the inverse projective Fourier transform, by decomposing $f(x) = \varphi^{\Omega}[L^{\dagger}_{\Omega}(x)\hat{f}]$, namely,

$$\begin{aligned} f(x) &= \sum_{\nu \in \mathbb{Z}-\{0\}} \mu(\nu)\, \mathrm{Tr}_\nu[S^{\dagger}_\nu(x)\hat{f}_\nu] \\ &\equiv \sum_{\nu \in \mathbb{Z}-\{0\}} \mu(\nu) f_\nu(x) \end{aligned} \tag{80}$$

Since $f_\nu(x) = \mathrm{Tr}_\nu[S^{\dagger}_\nu(x)\hat{f}_\nu] = f(x)$ for all ν [see (77)], the sum over the dual is, in fact, not needed and we have

$$f(x) = \mathrm{Tr}_\nu[S^{\dagger}_\nu(x)\hat{f}_\nu], \qquad \forall \nu \in \mathbb{Z} - \{0\} \tag{81}$$

The von Neumann algebra $\mathcal{M}^{\Omega}_\nu(\mathbb{R}^2)$ with the projective operator product of its generators $S_\nu(x)$, $x \in \mathbb{R}^2$, together with the trace Tr_ν and the remaining projective Kac algebra structure inherited from $\mathbb{K}^{\Omega}(\mathbb{R}^2)$, turns out to be a projective Kac algebra. This is so because, by property (77), the traces Tr_ν have the same characteristics of the trace φ^{Ω}, and are also Haar traces. This algebra will be denoted $\mathbb{K}^{\Omega}_\nu(\mathbb{R}^2)$, and its structure is given by

$$S_\nu(x)S_\nu(y) = e^{i\nu\Omega(x,y)}S_\nu(x+y) \tag{82a}$$

$$\mathbf{1} = S_\nu(0) \tag{82b}$$

$$\Delta^\nu S_\nu(x) = e^{i\nu\Theta(q;-x)}S_\nu(x) \otimes S_\nu(x) \tag{82c}$$

$$\kappa^\nu S_\nu(x) = e^{i\nu[\Theta(q;x)-\Theta(q;-x)]}S^{\dagger}_\nu(x) \tag{82d}$$

$$\mathrm{Tr}_\nu(T_\nu) = \begin{cases} \|f\|_2^2 & \text{if } T_\nu = \hat{f}^{\dagger}_\nu \cdot \hat{f}_\nu \\ +\infty & \text{otherwise} \end{cases} \qquad T_\nu \in \mathcal{M}^{\Omega}_\nu(\mathbb{R}^2)^+ \tag{82e}$$

These projective Kac algebras have exactly the same structure as that of $\mathbb{K}^{\Omega}(\mathbb{R}^2)$, so it is unnecessary to verify the axioms again. Observe that its GNS representation, as induced by Tr_ν, is in the Hilbert space $L^2(\mathbb{R}^2)$, while its elements act on $L^2(\mathbb{R})$. This is due to the fact that the operators

$$\hat{f}_\nu = \int_{\mathbb{R}^2} dx\, f(x) S_\nu(x) \tag{83}$$

are written in terms of $L^1_\Omega(\mathbb{R}^2)$ functions f, while the generators S_ν act on the "wavefunctions" on the configuration space.

The main difference between the above projective decomposition and the linear one performed in Aldrovandi and Saeger (1996) lies in the Haar weight decomposition. In the linear case the Haar weight (trace or not) does not satisfy (77) and is not, consequently, decomposed into Haar weights as happens with φ^Ω. In that case the Kac algebra decomposes into *Hopf–von Neumann* algebras generated by irreducible operators, and not into algebras of the same category (recall that a Kac algebra is just a Hopf–von Neumann algebra plus an n.f.s. Haar weight).

The predual of $\mathcal{M}^\Omega_\nu(\mathbb{R}^2)$ is obtained in the same way as in the previous case, that is, by duality. The representative functions $\hat{\omega}^\nu_{\xi\chi}$ in the predual are given by the pairing

$$\begin{aligned}\hat{\omega}^\nu_{\xi\chi}(x) &\equiv \langle S^\dagger_\nu(x), \hat{\omega}^\nu_{\xi\chi}\rangle = (S^\dagger_\nu(x)\xi \,|\, \chi)_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} dq\, e^{-i\nu\Theta(q;x)} \xi(q + x_1)\overline{\chi(q)}\end{aligned} \tag{84}$$

Also by duality, we obtain that the involution is conjugation by ^{o} and the product is the star product $\star$, operations already introduced in (59) and (58). The only difference between the present operations and those previously shown is a ν dependence in the phase factors. They are given explicitly by

$$f^o_\nu(x) = e^{-i\nu[\Theta(q;x)-\Theta(q;-x)]}\overline{f_\nu(x)}$$

$$(f_\nu \star g_\nu)(x) = e^{i\nu\Theta(q;x)} f_\nu(x) g_\nu(x)$$

where we have written the functions (84) as f_ν, g_ν, etc., to emphasize their ν dependence. From their definitions it follows also that these functions are essentially bounded, that is, they belong to $L^\infty_\Theta(\mathbb{R}^2)$ for every ν. The predual $\mathcal{M}^\Omega_\nu(\mathbb{R}^2)_*$ will be denoted $A^\Theta_\nu(\mathbb{R}^2)$ and can be interpreted as a ν-component of the projective Fourier algebra $A^\Theta(\mathbb{R}^2)$. With regard to Fourier representations, the ν-component $\hat{\sigma}_\nu$ of λ^Ω should be given by

$$[\hat{\sigma}_\nu(\hat{\omega}^\nu) f](q) = [(\hat{\omega}^\nu \circ \kappa^\nu \otimes id)\Delta^\nu \hat{f}_\nu]_{\mathrm{Tr}_\nu}, \qquad f = (\hat{f}_\nu)_{\mathrm{Tr}_\nu} \in L^1_\Omega \cap L^2(\mathbb{R}^2) \tag{85}$$

By the same kine of manipulations as made after (61), and recalling that $A^\Theta(\mathbb{R}^2)$ acts on $L^2(\mathbb{R}^2)$ by $\star$, the result is $\hat{\sigma}_\nu = id$. This should be interpreted as the injection of each $A_\nu^\Theta(\mathbf{R}^2)$ into the von Neumann algebra L_Θ^∞. The generator of this representation can be obtained from

$$(g \mid \hat{\sigma}_\nu(\hat{\omega}_{\xi\chi}^\nu) \star f)_{L^2(\mathbf{R}^2)} = (g \otimes \chi \mid W_\nu^\Theta(f \otimes \xi))_{L^2(\mathbf{R}^2) \otimes L^2(\mathbf{R})} \tag{86}$$

and turns out to be the operator in $A_\nu^\Theta(\mathbf{R}^2) \otimes \mathcal{M}_\nu^\Omega(\mathbf{R}^2)$ given by

$$W_\nu^\Theta(f, \xi)(x, q) = e^{i\nu[\Theta(q';x) - \Theta(q;x)]} f(x)\xi(q + x_1), \qquad q \neq q'$$

The phase factors come, respectively, from the $\star$-action on $L^2(\mathbf{R}^2)$ [q' comes from the product (67b)] and from the action of $S_\nu^\dagger(x)$ on $L^2(\mathbf{R})$. From the expression (62) for the fundamental operator W^Θ and from the fact that the action of $L_\Omega^\dagger(x)$ at y is decomposed into the action of $S_\nu^\dagger(x)$ at q, we verify that W_ν^Θ acts like $W_\nu^\Theta \sim 1 \otimes S_\nu^\dagger(x)$, and thus gives the genuine decomposition of W^Θ as the Fourier representation generator.

The coproduct and the coprojective coinvolution, when suitably decomposed from $\mathbf{K}^\Theta(\mathbf{R}^2)$, provide A_ν^Θ with the additional structure

$$\Delta_\nu f_\nu(x, y) = e^{-i\nu\Omega(x,y)} f_\nu(x + y)$$

$$\kappa_\nu f_\nu(x) = e^{-i\nu[\Theta(q;x) - \Theta(q;-x)]} f_\nu(-x)$$

In the same way in which W^Θ implements a coproduct, the generators W_ν^Θ implement the above coproducts and, as a consequence, also satisfy the pentagonal relation.

The predual of L_Θ^∞ has already been found: it is the non-Abelian algebra $L_\Omega^1(\mathbf{R}^2)$. Let us examine the decomposition of its Fourier representation λ^Θ. Since λ^Θ is, up to a restriction on its range of application, the left-regular representation of $L_\Omega^1(\mathbf{R}^2)$, its ν-component σ_ν should be given by (76) with a restriction in the range to $L_\Theta^\infty(\mathbf{R}^2) \cap L^2(\mathbf{R})$, that is,

$$\sigma_\nu(f) = \hat{f}_\nu = \int_{\mathbf{R}^2} dx\, f(x) S_\nu(x) \tag{87}$$

Needless to say, these are faithful involutive representations, mapping the twisted convolution into the projective operator product in $\mathbf{K}_\nu^\Omega(\mathbf{R}^2)$, for each ν in the projective dual. The generator of this representation is easily obtained from a formula analogous to (86) and is given by

$$W_\nu^\Omega(\xi, f)(q, y) = e^{i\nu[\Theta(q';-y) - \Theta(q;-y)]} \xi(q - y_1) f(y), \qquad q' \neq q$$

The same arguments which led us to recognize W_ν^Θ as the decomposition of W^Θ also lead us to identify W_ν^Ω as the irreducible decomposition of the dual W^Ω, for they behave like $W_\nu^\Omega \sim S_\nu(y) \otimes 1^o$. Furthermore, they also implement the coproducts (82c) and consequently satisfy the pentagonal relation.

Here ends our description of the projective duality decomposition.

6. WEYL QUANTIZATION AND DUALITY

We are now in a position to reexamine the Weyl–Wigner formalism in the context of the projective Fourier duality decomposition obtained in the last section. The expression for the projective Fourier transform (76) is formally equal to the expression (83) for the elements of $K_\nu^\Omega(\mathbb{R}^2)$, themselves given by the components σ_ν of the Fourier representation λ^Θ. It brings naturally to mind Weyl's formula, which associates a function on phase space to an irreducible projective operator on configuration space. Before proceeding to make this identification, we observe that, instead of the label ν, Weyl's formula exhibits the Planck constant $\hbar$ (Weyl, 1931, IV, §14). This fact leads us to consider a rescaling in the projective dual $\mathbb{Z} - \{0\}$ to $\hbar^{-1}\mathbb{Z} - \{0\}$, and to fix the value of the label ν as $\nu = 1$. Doing that means that we are selecting just one irreducible projective representation of $\mathbb{R}^2$ and just one projective Kac algebra $K_\hbar^\Omega(\mathbb{R}^2)$. This shows how quantum mechanics is restricted to a particular inequivalent representation or superselection sector (Landsman, 1993). In this context, Weyl's formula

$$\hat{f}_\hbar = \int_{\mathbb{R}^2} dx\, f(x) S_\hbar(x) \tag{88}$$

is a particular irreducible representation of L^1_Ω in the operator algebra $K_\hbar^\Omega(\mathbb{R}^2)$. The correspondence is completed when we write f in terms of this kind of operator. This follows from formula (81), which recovers f from (88) through

$$f(x) = \mathrm{Tr}_\hbar[S_\hbar^\dagger(x)\hat{f}_\hbar] \tag{89}$$

It is also possible to rewrite Weyl's formula as a linear combination of self-adjoint operators. This can be done by introducing operators $\tilde{S}_\hbar(y)$ such that the projective operators $S_\hbar(x)$ are their Fourier transforms:

$$S_\hbar(x) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} dy\, \overline{\chi_x(y)} \tilde{S}_\hbar(y) \tag{90}$$

where $\chi_x(y) = e^{(i/\hbar)xy}$. Comparing $S_\hbar^\dagger(x)$ and $S_\hbar(-x)$, we conclude that $\tilde{S}_\hbar^\dagger = \tilde{S}_\hbar$. When we substitute (90) in (88), we must also substitute the Fourier transform $\tilde{f}_\hbar$ for $f = f_\hbar \in L^1_\Omega$,

$$f(x) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} dz\, \chi_x(z)\, \tilde{f}_\hbar(z)$$

so that the two additional integrals are canceled out by the character-completeness relation

$$\int_{\mathbb{R}^2} dx\, \chi_x(z)\overline{\chi_x(z')} = (2\pi\hbar)^2 \delta(z - z')$$

The Weyl formula becomes

$$\hat{f}_\hbar = \int_{\mathbb{R}^2} dx\, \tilde{f}_\hbar(x) \tilde{S}_\hbar(x)$$

while the Fourier transform of (89) gives us back the function

$$\tilde{f}_\hbar(x) = \mathrm{Tr}_\hbar[\tilde{S}_\hbar(x)\hat{f}_\hbar]$$

As $\tilde{S}_\hbar$ is self-adjoint, this function is real. It is the Wigner *distribution function* associated to the operator $\hat{f}_\hbar$.

A particular distribution function is the Fourier transform of the $A_\hbar^\Theta(\mathbb{R}^2)$-function associated to the wavefunction ξ, which is given by (84) with $\chi = \xi$ and $\nu = \hbar^{-1}$. Changing variables in the integral, we can rewrite that formula as

$$\begin{aligned} \omega_{\xi\xi}^\hbar(x) &= (\xi \,|\, S_\hbar(x)\xi) \\ &= \int_{\mathbb{R}^2} dq\, e^{(i/\hbar)qx_2} \xi(q + x_1/2)\overline{\xi(q - x_1/2)} \end{aligned} \tag{91}$$

The Fourier transform of this function is just the Wigner distribution associated to the density operator $|\xi\rangle\langle\xi|$ (Hillery *et al.*, 1984; Lee, 1995),

$$W_\xi^\hbar(x) = [\mathcal{F}\omega_{\xi\xi}^\hbar](x) = \int_{\mathbb{R}} dq\, e^{-(i/\hbar)qx_1} \xi(x_2 + q/2)\overline{\xi(x_2 - q/2)}$$

Notice also that, if we match our notation with Dirac's through $(\chi \,|\, S\xi) = \langle\chi \,|\, S \,|\, \xi\rangle$, the function in (91) is the same L_Ω^1-function corresponding to the operator $\hat{f}_\hbar = |\xi\rangle\langle\xi|$, which is given by (89),

$$f^\xi(x) = \langle\xi \,|\, S_\hbar^\dagger(x) \,|\, \xi\rangle$$

The Wigner functions $\tilde{f}$ are also called "quantum" functions, since they depend on the constant $\hbar$. Since they are the Fourier transforms of $f \in L_\Omega^1$, they obey the noncommutative "twisted" product $\circ^\hbar$, the Fourier image of the twisted convolution

$$(f \circledast_\hbar g)(x) = \mathrm{Tr}_\hbar[S_\hbar^\dagger(x)\hat{f}_\hbar \cdot \hat{g}_\hbar]$$

which is explicitly given by

$$\begin{aligned} [\mathcal{F}(f \circledast_\hbar g)](z) &= \langle\overline{\chi_z}, f \circledast_\hbar g\rangle \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2\times\mathbb{R}^2} dx\, dy\, e^{(i/\hbar)\Omega(x,y)} \overline{\chi_z(x + y)} f(x) g(y) \\ &\equiv ([\mathcal{F}f] \circ^\hbar [\mathcal{F}g])(z) \end{aligned} \tag{92}$$

The algebra of these essentially bounded functions with the product $\circ^{\hbar}$ can be called the *Moyal algebra* and will be denoted here by $A_{\hbar}^{\Omega}(\mathbb{R}^2)$. The Fourier transform, which is a well-known isomorphism between the Abelian algebras L^1 and L^∞ over the plane, by (92) turns out to be also an isomorphism between the noncommutative algebras $L_{\Omega}^1(\mathbb{R}^2)$ and $A_{\hbar}^{\Omega}(\mathbb{R}^2)$ (Folland, 1989; Gracia-Bondía and Várilly, 1988). This isomorphism extends to $\mathcal{M}_{\hbar}^{\Omega}$ through M_{Ω}^1, for the generators $S_{\hbar}(x)$ are mapped [by (89)] into the "densities" $\delta_x \in M_{\Omega}^1$ and, by the Fourier transform, into the characters

$$\phi_x \equiv \frac{1}{2\pi\hbar} \overline{\chi_x}$$

which are the generators of the algebra $A_{\hbar}^{\Omega}$. We call ϕ_x the generators of that algebra because its elements are given by (the Fourier transforms)

$$\tilde{f}_{\hbar}(y) = \int_{\mathbb{R}^2} dx\, \phi_x(y) f(x) = [\mathcal{F}f](y)$$

Their product, according to (92), is given by

$$(\phi_x \circ^{\hbar} \phi_y)(z) = e^{(i/\hbar)\Omega(x,y)} \phi_{x+y}(z)$$

which proves the isomorphism of $\mathcal{M}_{\hbar}^{\Omega}$ and $A_{\hbar}^{\Omega}$. Extending it further (again through the Fourier transforms) to the Kac structure, we can add to the structure of $A_{\hbar}^{\Omega}$ the coproduct and the coinvolution of M_{Ω}^1 [see (57)]:

$$\begin{aligned}
[(\mathcal{F} \otimes \mathcal{F})\Delta^{\Omega}(f)](x, y) &= \langle \overline{\chi_x} \otimes \overline{\chi_y}, \Delta^{\Omega} f\rangle \\
&= \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} dz\, e^{(i/\hbar)\Theta(q;-z)} \overline{\chi_z(x + y)} f(z) \\
&\equiv [\Delta^{\hbar} \mathcal{F} f](x, y) \\
\mathcal{F}\kappa^{\Omega}(f)(x) &= \langle \overline{\chi_x}, \kappa^{\Omega}(f)\rangle \\
&= \frac{1}{2\pi\hbar} \int_{2} dy\, e^{(i/\hbar)[\Theta(q;y) - \Theta(q;-y)]} \chi_x(y) f(y) \\
&\equiv \kappa^{\hbar}(\mathcal{F} f)(x)
\end{aligned}$$

Furthermore, the involution is mapped into the complex conjugation and the unit into the constant function $\phi_0 = 1/(2\pi\hbar)$, while the Haar trace compatible with this structure is given by $\mathrm{Tr}^{\hbar}(\tilde{f}_{\hbar}) = \tilde{f}_{\hbar}(0)$. Summing up, the Kac structure of $A_{\hbar}^{\Omega}(\mathbb{R}^2)$ is given by

$$\begin{aligned}
\phi_x \circ^{\hbar} \phi_y &= e^{(i/\hbar)\Omega(x,y)} \phi_{x+y} \\
\mathbf{1} = \phi_0 &= 1/(2\pi\hbar) \\
\Delta^{\hbar} \phi_x &= e^{(i/\hbar)\Theta(q;-x)} \phi_x \otimes \phi_x \\
\kappa^{\hbar} \phi_x &= e^{(i/\hbar)[\Theta(q;-x) - \Theta(q;x)]} \overline{\phi_x} \\
\mathrm{Tr}^{\hbar}(\phi_x) &= \delta_x
\end{aligned}$$

7. FINAL REMARKS

To study how the Weyl–Wigner formalism inserts fits into the framework of general harmonic analysis, we have reviewed the role of the Heisenberg and the translation groups in the process of quantization on Euclidean phase space. Starting from a well-established (Fourier) duality for the Heisenberg group in terms of Kac algebras, we were able to introduce two new *projective Kac algebras*, in terms of which a *projective duality* for the translation group is defined. For these algebras to provide a projective duality, the usual coinvolution axioms have been suitably adapted to the projective framework, and this has forced us to introduce new operations. The irreducible decomposition of the symmetric projective Kac algebra according to the Ω-projective unitary dual of $\mathbb{R}^2$ was also performed, and it was shown how duality survives at the irreducible level. The preduality relations between whole and decomposed projective Kac algebras provide an explanation for the origin of the Weyl formula as an irreducible component of the Fourier representation of the Abelian projective Kac algebra. They also show the dual role played by the Weyl operators and respective quantum functions, where the latter are obtained from the first by Wigner's recovering formula and the Fourier transform. All these facts allow us to conclude that the Weyl–Wigner correspondence is incorporated in the projective (Fourier) duality of the translation group. We can go further and ask whether it is possible to generalize this duality principle to quantization on any other phase space. This question is partially answered in Aldrovandi and Saeger (1996), where the authors showed how far it is possible to extend this principle to the half-plane, whose canonical group, though requiring no central extension, has the awkward properties of being neither Abelian nor unimodular.

In the effort toward a general quantization prescription much has yet to be done. We have nevertheless, in the hard process of unraveling its pattern through case study and abstraction, obtained a glimpse of the basic frame and are in a position to risk a provisional proposal. Given a phase space, we should look for its linear canonical group. Find then its two Kac algebras, the symmetric and the Abelian. Examine the cohomology to see whether an extension is necessary, and proceed or not to it accordingly. The resulting symmetric algebra will be the space of quantum operators of the system.

ACKNOWLEDGMENTS

The authors would like to thank Prof. M. Enock for useful comments on the intricacies of Kac algebras. They also would like to thank CAPES and CNPq for financial support.

NOTE ADDED IN PROOF

The authors would like to thank Prof. R. L. Hudson for pointing out a work by him (Hudson, 1978) where he obtained similar results, although not departing from Kac algebras. It is worth mentioning that recently some related works departing from Kac algebras have appeared (Landstad and Raeburn, 1995; Enock and Vaĭnerman, 1996). Although they deal with projective representations, their objective and approach are quite different from ours.

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